Mock Exam is due tomorrow night 11:59 pm

Example Find $\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \lim_{x\to 0} \frac{(\cancel{1+x}+\frac{x^2}{2}+\frac{x^3}{6}+\cdots)\cancel{-1-x}}{x^2}$

$\boxed{e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots} = \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{6}+\frac{x^2}{4!}+\cdots\right)=\frac{1}{2}$

Example $\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2} = \lim_{x\to 0}\frac{\cancel{x}-(\cancel{x}-\frac{x^2}{2}+\frac{x^3}{3}-\cdots)}{x^2}$

$\boxed{\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots} = \lim_{x\to 0} +\frac{1}{2}-\frac{x}{3}+\frac{x^2}{4}+\cdots = \frac{1}{2}$

Example $\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}$

$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

$= \lim_{x\to 0}\frac{\cancel{1}-(\cancel{1}-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots)}{\cancel{1+x}-(\cancel{1+x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)}$

$= \lim_{x\to 0}\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots} = \lim_{x\to 0}\frac{\cancel{x^2}\left(\frac{1}{2}-\frac{x^2}{4!}+\cdots\right)}{\cancel{x^2}\left(-\frac{1}{2}-\frac{x}{3!}-\cdots\right)} = \frac{1/2}{-1/2} = 1$

Fall 2009 Final (Part B)

Part B

7. (25 points)

(a) Find the power series expansion of $1/(1+x^2)$, as well as radius and interval of convergence.

(b) Find the power series for $\arctan(x)$, as well as the radius and interval of convergence.

$\left(\frac{a}{1-r}\right)$ $a=1$ $r=-x^2$

$a+ar+ar^2+ar^3\cdots$

(a) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \underbrace{1-x^2+x^4-x^6+x^8+\cdots}_{\text{geometric series}}$

Only when $|r| = |-x^2|<1 \implies |x|<1=R$

Interval of Conv: $(-1,1)$

(b) $(\arctan(x))' = \frac{1}{1+x^2} \implies \arctan(x) = \int\frac{1}{1+x^2}dx$

$= \int 1-x^2+x^4-x^6+x^8+\cdots dx = C+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

$\arctan(0)=0 = C+0+0+\cdots$ $C=0$

$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ Radius of conv $=1$.

$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ $|x|<1$

$|x|=1 \Rightarrow x=\pm 1$

for $x=1$, $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ $b_n = \frac{1}{2n+1}$

$b_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3} \le \frac{1}{2n+1} = b_n$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n+1} = 0$

Conv Alternating Series Test

for $x=-1$, $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$

Interval of conv. $[-1,1]$

8. (25 points)

(a) Find the Taylor series centered at 0 of the function e^{-x^2}, as well as radius and interval of convergence.

(b) Write the integral

$$\int_0^x e^{-t^2}\,dt$$

as a power series.

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x

$R=\infty$

Interval of conv: $(-\infty,\infty)$

$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$

for all $-x^2$

$R=\infty$

I of C $=(-\infty,\infty)$

$\int_0^x e^{-t^2}dt = \int_0^x\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{n!}dt$ $R=\infty$

$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)\,n!}\Bigg|_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}$ $(-\infty,\infty)$

9. (15 points)

Find the radius and interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x+2)^n}{n+1}.$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}(x+2)^{n+1}}{n+2}\cdot\frac{n+1}{\cancel{(-1)^n}\cancel{(x+2)^n}}\right|$

$= \lim_{n\to\infty}\frac{n+1}{n+2}|x+2| = \lim_{n\to\infty}\frac{\cancel{n}(1+1/n)}{\cancel{n}(1+2/n)}|x+2| = |x+2|<1$

$|x-a|<R$

$R=1$

$a=-2$

$|x+2|=1 \Rightarrow x+2=\pm 1$

$x=-2\pm 1 = -3,-1$

for $x=-3$ $\sum_{n=0}^{\infty}\frac{(-1)^n(-3+2)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1} = \sum_{n=1}^{\infty}\frac{1}{n}$

DIV. (harmonic)

for $x=-1$ $\sum_{n=0}^{\infty}\frac{(-1)^n(-1+2)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ (alternating harmonic series)

CONV

Int. of Conv $= (-3,-1]$

10. (15 points)

Determine whether the series
$$\sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln(n)}$$
is absolutely convergent, conditionally convergent or divergent.

$\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n\ln n}\right| = \sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$ $\quad f(x)=\frac{1}{x\ln x} = (x\ln x)^{-1}$ $\quad x\geq 2$

f is cont, positive, decreasing since

$f'(x) = -(x\ln x)^{-2}\cdot(\ln x + x\cdot\frac{1}{x}) = -\frac{(\ln x+1)}{(x\ln x)^2}$ $\quad >0$ $\quad >0$

$f'(x)<0$ $\quad$ f is decreasing ✓

$\int_2^{\infty}\frac{1}{x\ln x}dx \qquad u=\ln x \qquad du=\frac{1}{x}dx$

$=\lim_{t\to\infty}\int_2^t\frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u}du = \lim_{t\to\infty}\ln|u|\Big|_{\ln 2}^{\ln t}$

$=\lim_{t\to\infty}\ln|\underbrace{\ln t}_{\infty}| - \underbrace{\ln|\ln 2|}_{C} = \infty$ DIV $\to \infty$

Thus, by the integral test, $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ is DIV.

Meaning: $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is not ABS CONV.

$b_n=\frac{1}{n\ln n}=f(n)$ $\quad$ f is decreasing and therefore b_n is decreasing. ($b_{n+1}\leq b_n$) ✓

$\lim_{n\to\infty}\frac{1}{n\ln n}=0$ ✓ $\quad$ Thus, $\sum\frac{(-1)^n}{n\ln n}$ is CONV and Conditionally Conv. (since it is not ABS CONV)

11. (20 points)

(a) Determine whether the series
$$\sum_{n=0}^{\infty}(-1)^n e^{-n}$$
is absolutely convergent, conditionally convergent or divergent.

(both the ratio and root tests should work)

(b) Estimate the sum of the series within an accuracy of e^{-5}. You may leave you answer in terms of powers of e.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}e^{-(n+1)}}{\cancel{(-1)^n}e^{-n}}\right| = |e^{-1}| = \frac{1}{e}<1$

By the ratio test, $\sum(-1)^n e^{-n}$ is ABS. CONV.

$\sum_{n=0}^{\infty}(-1)^n\underbrace{e^{-n}}_{\text{always positive}}$ is an alternating series.

$\boxed{|R_n|\leq b_{n+1} = |a_{n+1}| = e^{-(n+1)}\leq e^{-5}}$

$-(n+1)\leq -5$

$-n-1\leq -5$

$n+1\geq 5 \qquad \boxed{n\geq 4} \qquad n=4$

$\sum_{n=0}^{4}(-1)^n e^{-n} = 1-\frac{1}{e}+\frac{1}{e^2}-\frac{1}{e^3}+\frac{1}{e^4}$