11.10 Taylor and Maclaurin Series

$\left(\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x|<1 \right)$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \qquad |x-a|<R$

$f(a) = c_0 \longrightarrow c_0 = \frac{f(a)}{0!} = \frac{f^{(0)}(a)}{0!}$ $\quad \boxed{0! = 1}$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3$

$f'(a) = c_1 \longrightarrow c_1 = \frac{f'(a)}{1!}$

$f''(x) = 2c_2 + 3\cdot 2 c_3(x-a) + 4\cdot 3 c_4(x-a)^2$

$f''(a) = 2c_2 \rightarrow c_2 = \frac{f''(a)}{2!}$

$f'''(x) = 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + \cdots$

$f'''(a) = 3\cdot 2 c_3 \rightarrow c_3 = \frac{f'''(a)}{3!}$

$c_n = \frac{f^{(n)}(a)}{n!}$

Thm If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad |x-a|<R$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Thus,

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \leftarrow$ Taylor series of f at a

If $a=0$, $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \leftarrow$ Maclaurin series of f.

Example Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence.

$f^{(0)}(x)=e^x \quad f'(x)=e^x \quad f''(x)=e^x \ \cdots \quad f^{(n)}(x)=e^x$

$f^{(0)}(0)=e^0=1 \quad f'(0)=1 \quad f''(0)=1 \ \cdots \quad f^{(n)}(0)=e^0=1$

Thus, $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\cancel{x^{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right| = \frac{|x|}{n+1} \to 0 \quad \text{as } n\to\infty$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, the series CONV for all x by the Ratio test. $R=\infty$

Q) How can we determine whether e^x does have a power series representation?

(More generally, how about $f(x)$?)

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$\boxed{n\text{th degree Taylor polynomial of } f \text{ at } a}$

Q) Is $\lim_{n\to\infty} T_n(x) = f(x)$?

$R_n(x) = f(x) - T_n(x) \qquad$ Is $\lim_{n\to\infty} R_n(x) = 0$?

Thm If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0 \quad |x-a|<R$

then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

Thm (Taylor's Inequality)

If $|f^{(n+1)}(x)| \le M$ for $|x-a|\le d$, then the remainder $R_n(x)$ satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a|\le d$$

Example Prove that e^x is equal to the sum of its Maclaurin series. $\quad \boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0}$

$\boxed{\text{We want to show that } \lim_{n\to\infty} R_n(x) = 0 \text{ on some interval}}$

$f(x)=e^x \quad f'(x)=e^x \ \cdots \quad f^{(n)}(x)=e^x \quad$ Thus

$|f^{(n+1)}(x)| \le \underbrace{e^d}_{=M}$ for $|x|\le d \quad [-d, d]$

Thus, $|R_n(x)| \le \dfrac{M \cdot |x-a|^{n+1}}{(n+1)!} = \dfrac{e^d |x|^{n+1}}{(n+1)!} \to 0$ $(a=0)$

$$\lim_{n\to\infty} \frac{e^d |x|^{n+1}}{(n+1)!} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

Thus, $\lim_{n\to\infty} |R_n(x)| = 0$ by the Squeeze theorem.

Therefore, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $|x| \le d$

Since this argument works for arbitrary d,

in fact we have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x in $(-\infty, \infty)$.

(Plug in $x=1$; $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$)

Example Find the Maclaurin series for $\sin x$. $(a=0)$

$f^{(0)}(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$ $\quad f'''(x) = -\cos x$ $\quad f^{(4)}(x) = \sin x$

$f^{(0)}(0) = 0$ $\quad f'(0) = 1$ $\quad f''(0) = 0$ $\quad f'''(0) = -1$ $\quad 0$

$$\sin x \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + \frac{(1)x}{1!} + \frac{0x^2}{2!} + \frac{(-1)x^3}{3!} + 0 + \frac{(1)x^5}{5!} + \cdots$$

$$\sin x \overset{?}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$|f^{(n+1)}(x)| \le M = 1$ $\qquad |R_n(x)| \le \dfrac{1 \cdot |x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$

$\pm \sin x$
$\pm \cos x$

Thus, $|R_n(x)| \to 0$ by the Squeeze thm.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for all } x.$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x$$

Example Find the Maclaurin series for $\cos x$.

Since $(\sin x)' = \cos x$

$$\cos x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)'$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example Find the Maclaurin series for $f(x) = x\cos x$

$$x\cos x = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$$

Example Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\pi/3$.

$f^{(0)}(x) = \sin x$ $\quad f'(x) = \cos x$ $\quad f''(x) = -\sin x$ $\quad f'''(x) = -\cos x$ $\quad f^{(4)} = \sin x$

$f^{(0)}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\quad -\frac{\sqrt{3}}{2}$ $\quad -\frac{1}{2}$

The Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} \frac{\left(x-\frac{\pi}{3}\right)^2}{2!} - \frac{1}{2} \frac{\left(x-\frac{\pi}{3}\right)^3}{3!} + \cdots$$

Example Find the Maclaurin series for $f(x) = (1+x)^k$

$f^{(0)}(x) = (1+x)^k \xrightarrow{x=0} f^{(0)}(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad f''(0) = k(k-1)$

$\vdots$ $\qquad \vdots$

$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$ $\qquad f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

"binomial series"

Notation: $\binom{k}{n} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$ called the "binomial coefficients"

Thus, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \binom{k}{n} x^n = (1+x)^k \quad |x| < 1}$

Example Find the Maclaurin series for $f(x) = \dfrac{1}{\sqrt{4-x}}$

$$f(x) = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2} \cdot \frac{1}{\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\boxed{\left(1-\frac{x}{4}\right)^{-1/2}} \qquad \left(1 + \left(\frac{-x}{4}\right)\right)^k$$

$x \to \frac{-x}{4}$ $\qquad k = -\frac{1}{2}$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{-x}{4}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!} \cdot \frac{(-1)^n x^n}{4^n}$$