Example For what values of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ CONV?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{\cancel{(x-3)^n}}\right| = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}|x-3|\right)$$

$= |x-3|$ if $|x-3|<1$ then CONV
if $|x-3|>1$ then DIV.

What about $|x-3|=1$? $|x-3|=1 \to x-3=\pm 1$

$x = 3\pm 1$ $x = 2,4$.

For $x=2$, we get

$$\sum_{n=1}^{\infty}\frac{(2-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ CONV ("alternating harmonic series")

For $x=4$,

$$\sum_{n=1}^{\infty}\frac{(4-3)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$ DIV ("harmonic series")

Thus, The series CONV on $[2,4)$ Final answer

$(|x-3|<1 \to -1<x-3<1 \to 2<x<4)$
$(2,4)$

Theorem For $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities

1) The series CONV only when $x=a$
2) The series CONV for all x
3) There is a positive constant R such that the series CONV if $|x-a|<R$ and diverges $|x-a|>R$

R: "radius of convergence"

The interval on which the series converges is called "the interval of convergence"

We say $R=0$ in case 1
and $R=\infty$ in case 2

Example Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$ $(x-a)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}\cdot\frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \lim_{n\to\infty}\frac{|x+2|}{3}\left(\frac{n+1}{n}\right) = \frac{|x+2|}{3}$$ $\left(\frac{n+1}{n}\to 1\right)$

$(a=-2)$

$\frac{|x+2|}{3}<1 \Rightarrow |x+2|<3$ $|x-a|<R$

$R=3$

If $|x+2|=3 \Rightarrow x+2=\pm 3$ $x=-2\pm 3$
$x=-5,1$.

For $x=-5$,

$$\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n(-1)^n\cancel{3^n}}{\cancel{3^{n+1}}\,3} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{3}$$

a_n is DIV

For $x=1$

$$\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n\cancel{(3)^n}}{\cancel{3^{n+1}}\,3} = \sum_{n=0}^{\infty}\frac{n}{3}$$ $a_n\to\infty$

In both case the series is DIV by the test for divergence

The interval of Conv is $I=(-5,1)$.

11.9 Representations of Functions as Power Series

$\left(\frac{a}{1-r}\right.$ $|r|<1)$ $|x|<1$ $I=(-1,1)$ $R=1$

$$\frac{1}{1-x} \overset{\circledast}{=} \underbrace{1+x+x^2+x^3+x^4+\cdots}_{\text{geometric series } |r|<1} = \sum_{n=0}^{\infty}x^n$$

$|x|<1$

Example Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of Conv.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1-x^2+x^4-x^6+x^8+\cdots$$
$$= \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$|-x^2|<1$

$|x^2|<1$ $R=1$

$|x|<1 \longrightarrow I=(-1,1)$

Example Find a power series representation for $\frac{1}{x+2}$.

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\cdot\frac{1}{1-(-\frac{x}{2})}$$

$$= \frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{2^2}-\frac{x^3}{2^3}+\frac{x^4}{2^4}-\cdots\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$\left|-\frac{x}{2}\right|<1$ $|x-a|<R$ $(a=0)$ $= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$

$\frac{|x|}{2}<1 \to |x|<2$ $R=2$ $I=(-2,2)$

Example $\dfrac{x^3}{x+2} = x^3\left(\dfrac{1}{x+2}\right)$

$$= x^3\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{2^{n+1}}$$

Theorem If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of conv $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

1) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

2) $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

In both cases, the radius of convergence is still R.

Example Express $\dfrac{1}{(1-x)^2}$ as a power series using (*)

$$\frac{1}{1-x} \overset{(*)}{=} 1 + x + x^2 + x^3 + \dots \qquad R=1$$

Notice that $\left(\dfrac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = -1(1-x)^{-2}\cdot(-1)$

$$= \frac{1}{(1-x)^2}$$

Thus if we take the derivative of both sides of (*), we get

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$$

$R=1$

Example Find a power series representation for $\ln(1+x)$ and its radius of conv.

$$(\ln(1+x))' = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \dots \qquad |x|<1 \quad R=1$$

Integrating both sides,

$$\int(\ln(1+x))'dx = \ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Plug in $x=0$

$$0 = \ln(1) = C + 0 + 0 + \dots \qquad C=0$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R=1$$

(or $x \to x-1$: $\ln(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^3}{4} + \dots = \sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(x-1)^n}{n}$)

Example Find a power series representation for $f(x) = \tan^{-1}x$.

$$(\tan^{-1}x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad R=1$$

$$\downarrow$$

$$\tan^{-1}x = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$\tan^{-1}(0) = 0 = C + 0 - 0 + \dots \qquad C=0$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$R=1$

$$\boxed{\begin{array}{l} \frac{1}{1-A} = 1 + A + A^2 + A^3 \dots \quad |A|<1 \\ A = -x^2 \to |x^2|<1,\ |x|<1 \\ \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \dots \end{array}}$$

Example Evaluate $\int \dfrac{1}{1+x^7}dx$ as a power series

Approximate $\int_0^{0.5}\dfrac{1}{1+x^7}dx$ correct to within 10^{-7}.

$$\frac{1}{1+x^7} \overset{R=1}{=} 1 - x^7 + x^{14} - x^{21} + \dots = \sum_{n=0}^{\infty}(-1)^n x^{7n}$$

$$|x^7|<1 \iff |x|<1 \quad R=1$$

$$\int\frac{1}{1+x^7}dx = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{7n+1}}{7n+1}$$

$$\int_0^{0.5}\frac{1}{1+x^7}dx = \left(C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22}\right)\Big|_0^{0.5}$$

$$= \left(\cancel{C} + 0.5 - \frac{0.5^8}{8} + \frac{0.5^{15}}{15} - \frac{0.5^{22}}{22} + \dots\right) - \left(\cancel{C} + 0 + 0 \dots\right)$$

$$= 0.5 - \frac{0.5^8}{8} + \frac{0.5^{15}}{15} - \frac{0.5^{22}}{22} + \dots \qquad |\text{Error}| < 10^{-7}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(0.5)^{7n+1}}{7n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{7n+1}(7n+1)} \qquad \frac{1}{2^{7n+1}(7n+1)} < 10^{-7}$$

$|\text{Error}| = |R_n| = b_{n+1} < 10^{-7}$ for $n=4$ we get $\dfrac{1}{2^{29}(29)} < 10^{-7}$

So it is enough to use the first three terms.