The final exam sign-up will open this Friday.

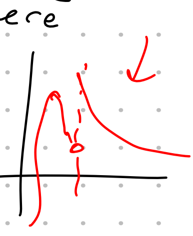
The final exam will be 90 minutes long.

More Examples from Chp 11

Example $\sum_{n=1}^{\infty} n e^{-n^2}$ $\to$ $\frac{x}{e^{x^2}} = xe^{-x^2} \to \int_1^{\infty} xe^{-x^2}\,dx$ is easy ($u=-x^2$)

$\to \sum \frac{n}{e^{n^2}}$

so we can use the integral test here



or we can use the ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n}$

$= \lim_{n\to\infty} \frac{n+1}{n} \frac{\cancel{e^{n^2}}}{e^{\cancel{n^2}+2n+1}} = \lim_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)}_{1} \cdot \underbrace{\frac{1}{e^{2n+1}}}_{0} = 0 < 1$

Thus, $\sum n e^{-n^2}$ is CONV by the ratio test.

Example $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$ $\leadsto$ $\left|(-1)^n \frac{n^3}{n^4+1}\right| = \frac{n^3}{n^4+1} \overset{LCT}{\sim} \frac{n^3}{n^4} = \frac{1}{n}$

not useful because $\sum |a_n|$ is DIV; $\sum \frac{1}{n}$ is DIV

$\frac{n^3}{n^4+1} \le \frac{1}{n}$, DIV$(\sum a_n)$, DIV$(\sum a_n)$

$b_n = |a_n| = \frac{n^3}{n^4+1}$

- $b_{n+1} \le b_n$ ✓
- $\lim_{n\to\infty} b_n = 0$ ✓

$f(x) = \frac{x^3}{x^4+1}$ $\quad f'(x) = \frac{3x^2(x^4+1) - x^3(4x^3)}{(x^4+1)^2}$

$= \frac{3x^6+3x^2-4x^6}{(x^4+1)^2} = \frac{-x^6+3x^2}{(x^4+1)^2} = \frac{x^2(-x^4+3)}{(x^4+1)^2}$

Thus by the AST $\sum (-1)^n \frac{n^3}{n^4+1}$ is CONV.

for $x > 2$: $f'(x) < 0$, f is decreasing

Example $\sum_{k=1}^{\infty} \frac{2^k}{k!} \leadsto \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\cancel{2^{k+1}}\,2}{\cancel{(k+1)!}\,(k+1)} \cdot \frac{\cancel{k!}}{\cancel{2^k}}$

$= \lim_{k\to\infty} \frac{2}{k+1} = 0 < 1$ by the ratio test, $\sum \frac{2^k}{k!}$ is CONV.

Example $\sum_{n=1}^{\infty} \frac{1}{2+3^n} \overset{LCT}{\sim} \sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ is a geometric series with $|r| = \frac{1}{3} < 1$ so it is CONV.

This is also CONV.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2+3^n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{3^n}\left(\frac{2}{3^n}+1\right)}}{\frac{1}{\cancel{3^n}}} = \lim_{n\to\infty} \frac{1}{\underbrace{\frac{2}{3^n}}_{0}+1} = 1 \neq 0$

so LCT does apply.

Example $\sum \frac{n^{2n}}{(1+n)^{3n}} = \sum \frac{(n^2)^n}{((1+n)^3)^n}$

$= \sum \left(\frac{n^2}{(1+n)^3}\right)^n$

$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\left(\frac{n^2}{(1+n)^3}\right)^{\cancel{n}}\right)^{\cancel{1/n}} = \lim_{n\to\infty} \frac{n^2}{(1+n)^3} = \lim_{n\to\infty} \frac{n^2}{\left(n\left(\frac{1}{n}+1\right)\right)^3}$

$= \lim_{n\to\infty} \frac{1\cancel{n^2}}{n\cancel{n^2}\left(\frac{1}{n}+1\right)^3} = \lim_{n\to\infty} \frac{1}{\underbrace{n}_{\infty}\underbrace{\left(\frac{1}{n}+1\right)^3}_{1}} \to \infty \Big] \to 0 < 1$

Thus, $\sum \frac{n^{2n}}{(1+n)^{3n}}$ is CONV by the root test.

Example $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n} \le \cancel{\sum_{n=1}^{\infty} \frac{1}{1+2^n}}$ $\qquad -1 \le \sin 2n \le 1$

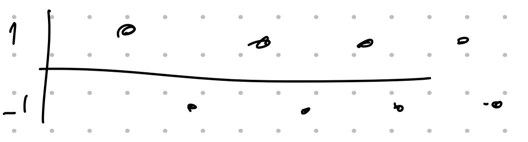
$0 \le |\sin 2n| \le 1$

$\sum_{n=1}^{\infty} \frac{|\sin 2n|}{1+2^n} \le \sum_{n=1}^{\infty} \frac{1}{1+2^n} \overset{LCT}{\sim} \sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$

CONV $\leftarrow$ Direct C.T. $\leftarrow$ CONV $\leftarrow$ Geom. w/ $|r| = \frac{1}{2} < 1$ CONV

Thus $\sum \frac{\sin 2n}{1+2^n}$ is ABS. CONV $\to$ it is also CONV.

Example $\sum (-1)^n \cos\left(\frac{1}{n^2}\right)$



as $n\to\infty$ $\quad b_n = \cos\left(\underbrace{\frac{1}{n^2}}_{0}\right) \to \cos 0 = 1 \neq 0$

$a_n = (-1)^n b_n$ is DIV

(Test for div: $\sum a_n$ is DIV if $\lim_{n\to\infty} a_n \neq 0$)

$\sum a_n$ is DIV by the test for divergence

11.8 Power Series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

c_n are called the coefficients.

Define $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ Domain is the set of all x for which the series is CONV.

e.g. let $c_n = 1$: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ geometric series is conv $|x| < 1$ and it diverges if $|x| \ge 1$

thus, the domain of $f(x) = \sum_{n=0}^{\infty} x^n$ is $(-1, 1)$.

$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$

More generally, $\sum_{n=0}^{\infty} c_n (x-a)^n$ is called a power series centered at a or power series in $(x-a)$.

Example For what values of x is the series $\sum_{n=0}^{\infty} n!\, x^n$ CONV?

The ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\cancel{(n+1)!}\,\cancel{|x|^{n+1}}}{\cancel{n!}\,\cancel{|x|^n}} = \lim_{n\to\infty}(n+1)|x| \qquad \lim_{n\to\infty}\overset{\infty}{(2n)}\left(\overset{0}{\frac{1}{n}}\right)=2 \qquad (n+1)\cdot 0 = 0$$

if $\boxed{|x|=0}$ (or $x=0$), then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0<1$ by the ratio test this is CONV.

if $|x|\neq 0$, then $\lim_{n\to\infty}(n+1)|x| = \infty$ by the ratio test, this is DIV.

So $\sum_{n=0}^{\infty} n!\,x^n$ is CONV if and only if $x=0$

$$\underbrace{0!}_{0!=1} + \underbrace{1!\,x + 2!\,x^2 + \cdots}_{0}$$

Example For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$ CONV?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-3|^{\cancel{n+1}}}{n+1}\cdot\frac{n}{\cancel{|x-3|^n}} = \lim_{n\to\infty}|x-3|\,\frac{n}{n+1}$$

$$= \lim_{n\to\infty}|x-3|\frac{\cancel{n}}{\cancel{n}(1+\underbrace{\tfrac{1}{n}}_{0})} = |x-3|\cdot\frac{1}{1+0} = |x-3|$$

if $|x-3|<1$, then $\sum\frac{(x-3)^n}{n}$ is CONV

and if $|x-3|>1$, then $\sum\frac{(x-3)^n}{n}$ is DIV by the ratio test

Q) $|x-3|=1$? Next time!