**Recall:** $\sum a_n$, $\quad \sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$

**Def$^n$** If $\sum |a_n|$ is CONV then $\sum a_n$ is called <u>absolutely convergent</u> (ABS CONV)

**Example** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ this is CONV ($p$-series with $p > 1$)

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is ABS CONV.

(It can be shown that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is CONV using the alternating series test)

**Example** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is CONV using the alternating series test

- $b_n = |a_n| = \frac{1}{n}$
- $b_{n+1} \le b_n$ ✓
- $\lim_{n\to\infty} b_n = 0$ ✓

However, $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is DIV ($p$-series with $p \le 1$)

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is <u>not ABS CONV</u>.

**Def$^n$** A series $\sum a_n$ is called <u>conditionally convergent</u> if it is CONV but not ABS CONV.

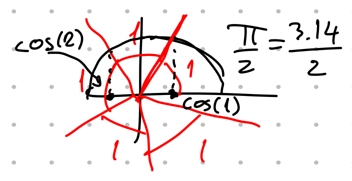
So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is <u>COND. CONV</u>.

**Thm** If a series $\sum a_n$ is ABS CONV then it is CONV.

**Example** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is CONV or DIV




We consider

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \qquad |\cos n| \le 1 \to \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ is CONV ($p$-series $p>1$), by the direct comparison test $\sum \frac{|\cos n|}{n^2}$ is also CONV.

Thus, $\sum \frac{\cos n}{n^2}$ is ABS. CONV and therefore CONV.

**The Ratio Test**

Consider $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

i) If $L < 1$, then $\sum a_n$ is ABS CONV (and therefore CONV)

ii) If $L > 1$ or $L = \infty$ then $\sum a_n$ is DIV

iii) If $L = 1$, then the ratio test is <u>inconclusive</u>.

**Note:** Consider $\sum \frac{1}{n^2}$ which is <u>CONV</u>.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left( \frac{n}{n+1} \right)^2 = \lim_{n\to\infty} \left( \frac{1}{1+\frac{1}{n}} \right)^2 = 1$$

Consider $\sum \frac{1}{n}$ which is <u>DIV</u>.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

**Example** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ $\qquad |a_{n+1} \cdot a_n^{-1}|$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{\cancel{3^{n+1}}_{3}} \cdot \frac{\cancel{3^n}}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{3} \left( \frac{n+1}{n} \right)^3 = \lim_{n\to\infty} \frac{1}{3} \left( \frac{\frac{1}{n}(n+1)}{\frac{1}{n}n} \right)^3 = \lim_{n\to\infty} \frac{1}{3} \left( 1 + \frac{1}{n} \right)^3 = \frac{1}{3} < 1$$

Thus, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is ABS. CONV (and therefore CONV)

**Example** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\qquad a_{n+1} \cdot a_n^{-1}$

$\boxed{\begin{array}{l} n! = 1 \cdot 2 \cdot 3 \cdots n \\ (n+1)! = 1 \cdot 2 \cdots n \cdot (n+1) \\ \quad = n!\,(n+1) \end{array}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n\cancel{+1}}}{\cancel{n+1}} \cdot \frac{1}{n^n} = \lim_{n\to\infty} \left( \frac{n+1}{n} \right)^n = \lim_{n\to\infty} \left( 1 + \frac{1}{n} \right)^n = e > 1$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is DIV by the ratio test.

$\displaystyle\lim_{n\to\infty} \left( 1 + \frac{1}{n} \right)^n \quad (1^\infty) \qquad A = e^{\ln A}$

$$= \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} e^{n \ln\left(1+\frac{1}{n}\right)} = e^{\left(\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)\right)}$$

$$= e^{\left(\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)} \overset{L'H}{=} e^{\left(\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}\right)} = e^1 = e$$

2nd method $\frac{n^n}{n!} = \frac{n\cdot n\cdot n\cdot n \cdots n}{1\cdot 2\cdot 3\cdots n} \geq n \to \infty$ as $n\to\infty$

Thus, $\frac{n^n}{n!}\to\infty$ and $\sum \frac{n^n}{n!}$ is DIV (test for divergence)

## The Root Test

Consider $\lim\limits_{n\to\infty} |a_n|^{1/n} = L$

i) If $L<1$ then $\sum a_n$ is ABS CONV (and therefore CONV)

ii) If $L>1$ or $L=\infty$ then $\sum a_n$ is DIV

iii) If $L=1$ then the root test is inconclusive.

**Example** $\sum\limits_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\left(\frac{2n+3}{3n+2}\right)^{\cancel{n}}\right)^{\cancel{1/n}} = \lim_{n\to\infty} \frac{2n+3}{3n+2}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$$ so it is ABS. CONV.

## 11.7 Examples from Chp 11 (so far)

**Example** $\sum\limits_{n=1}^{\infty} \frac{n-1}{2n+1}$ Notice that $\frac{n-1}{2n+1} = \frac{1-\frac{1}{n}}{2+\frac{1}{n}} \to \frac{1}{2}$

By the test for divergence $\sum a_n$ is DIV.

**Example** $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ $\qquad b_n = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3-3/2}} = \frac{1}{n^{3/2}}$

Limit Comp.: $b_n^{-1} \cdot a_n$ $\qquad \sum b_n = \sum \frac{1}{n^{3/2}}$ CONV (p-series $p>1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}\,(n^3+1)^{1/2}}{3n^3+4n^2+2}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\left[n^3\left(1+\frac{1}{n^3}\right)\right]^{1/2}}{n^3\left(3+\frac{4}{n}+\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{1}{n^3}\right)^{1/2}}{\cancel{n^3}\left(3+\frac{4}{n}+\frac{2}{n^3}\right)} = \frac{1}{3} \neq 0$$

By the Limit Comparison T. $\sum a_n$ CONV since $\sum b_n$ CONV.