Midterm 3 is next week (Thursday/Friday)

Sign-up form will be emailed today!

Recall: $\sum a_n$, $\sum b_n$ with positive terms

and $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$ (finite) then

either both $\sum a_n$ and $\sum b_n$ CONV or both DIV.

Example Determine whether the series $\sum_{n=1}^{\infty} \underbrace{\frac{2n^2+3n}{\sqrt{5+n^5}}}_{a_n}$ CONV or DIV.

$$b_n = \frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^{5/2}} = \frac{1}{n^{5/2-2}} = \frac{1}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} n^{1/2}\left(\frac{2n^2+3n}{(5+n^5)^{1/2}}\right)$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{(5+n^5)^{1/2}} = \lim_{n\to\infty} \frac{n^{5/2}(2+3/n)}{(n^5(\frac{5}{n^5}+1))^{1/2}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^{5/2}}(2+\underbrace{3/n}_{\to 0})}{\cancel{n^{5/2}}(\underbrace{\frac{5}{n^5}}_{\to 0}+1)^{1/2}} = \frac{2+0}{(0+1)^{1/2}} = 2 \neq 0$$ and it's finite

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is DIV (p-series with $p \leq 1$)

Therefore by the limit comparison test $\sum_{n=1}^{\infty} a_n$ is also DIV.

Estimating Sums

Say $\sum a_n$, $\sum b_n$ a_n, b_n are both positive and

$\boxed{a_n \leq b_n}$ (So we can use the (direct) comparison test)

Say $\sum_{n=1}^{\infty} a_n = s$ and $\sum_{n=1}^{\infty} b_n = t$

$$R_n = s - \sum_{i=1}^{n} a_i \quad \leq \quad T_n = t - \sum_{i=1}^{n} b_i$$

$$= a_{n+1} + a_{n+2} + a_{n+3} + \cdots \qquad = b_{n+1} + b_{n+2} + b_{n+3} + \cdots$$

Example Say we use the sum of the first 100 terms to approximate the sum of the series $\sum \frac{1}{(n^3+1)}$. Estimate the error involved in this approximation.

Remainder estimate for the integral test

$$\frac{1}{n^3+1} \leq \frac{1}{n^3} \longrightarrow T_n \leq \int_n^{\infty} \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left. \frac{-x^{-2}}{2} \right|_n^t = \frac{1}{2n^2}$$

Thus $R_{100} \leq T_{100} \leq \frac{1}{2(100)^2} = 0.00005$

11.5 Alternating Series

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

$a_n = (-1)^{n-1} b_n$ or $a_n = (-1)^n b_n$ where b_n is positive.

In fact, $b_n = |a_n|$

Alternating Series Test If the alternating series

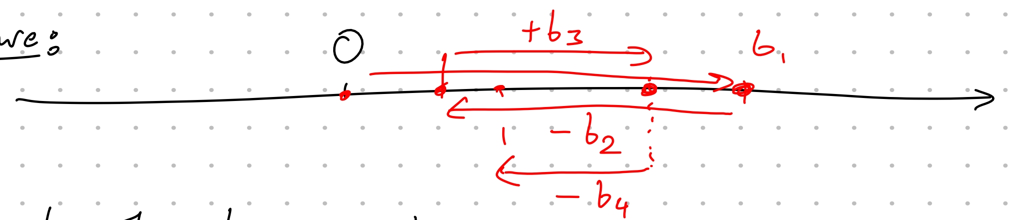
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \qquad b_n > 0$$

satisfies (i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is CONV.

Picture:



Example The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$a_n = \frac{(-1)^{n-1}}{n}$ $b_n = |a_n| = \frac{1}{n}$

$b_{n+1} < b_n$ decreasing

$b_{n+1} \leq b_n$ non-increasing

$b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$ ✓

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

Thus, by the alternating series test the series is CONV.

Example The series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating.

However, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-1/n} = \frac{3}{4} \neq 0$ ✗

(The alternating series test does not apply)

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n b_n$ DIV. Thus, $\sum_{n=1}^{\infty} a_n$ is DIV by the Test for Divergence.

Example $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

$b_n = \frac{n^2}{n^3+1}$ $\quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{n^3}\, n^2}{\frac{1}{n^3}(n^3+1)}$

$= \lim_{n\to\infty} \frac{\frac{1}{n} \to 0}{1+\frac{1}{n^3} \to 0} = 0$

Q) How to check if $b_{n+1} \le b_n$?

Set $f(x) = \frac{x^2}{x^3+1}$ so that $b_n = f(n)$

$f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{2x^4 + 2x - 3x^4}{(x^3+1)^2}$

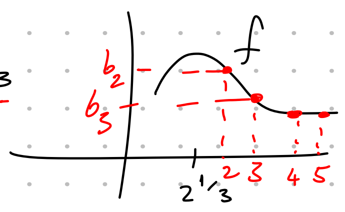
$= \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$ negative $\quad (x > 2^{1/3})$

So f is decreasing for $x > 2^{1/3}$

Thus $f(n+1) < f(n)$

$\quad\; \| \qquad\quad \|$

$\; b_{n+1} < b_n$



Thus by the alternating series test $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ is CONV.

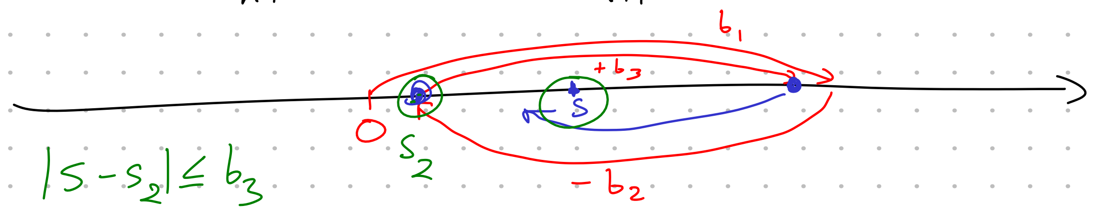
Estimating Sums

Alternating Series Estimation Theorem

If $s = \sum (-1)^{n-1} b_n$, where $b_n > 0$ and

(i) $b_{n+1} \le b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$



$|s - s_2| \le b_3$

Example How many terms are needed in approximating the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ with error less than 0.01.

$\left(= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots \right)$

$|\text{Error}| = |R_n| \le b_{n+1} = \frac{1}{(n+1)!} \le 0.01 = \frac{1}{100}$ $\qquad b_n = \frac{1}{n!}$

$\frac{1}{(n+1)!} \le \frac{1}{100}$ $\qquad (n+1)! \ge 100$

$\qquad 5! = 120 \qquad n \ge 4$

11.6 Absolute Convergence and the Ratio and the Root Tests

Given $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

consider $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$

Def^n A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is CONV.