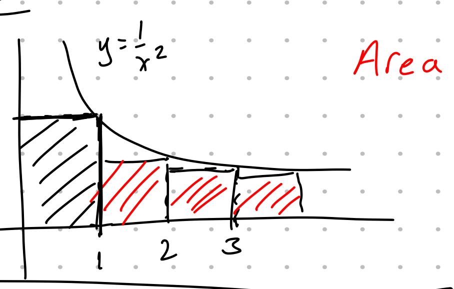
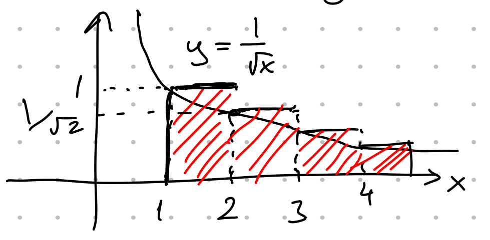
Recall:




$$\text{Area} = \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

thus the sequence $\{s_n\}$ of partial sums is a bounded monotonic seq. and it is CONV by the monotonic sequence theorem.

Similarly, for $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ we consider the following picture




$$\sum_{n=1}^{t} \frac{1}{\sqrt{n}} \geq \int_1^{t} \frac{1}{\sqrt{x}}\,dx$$

$\searrow \infty$ $\qquad \searrow \infty$ as $t \to \infty$

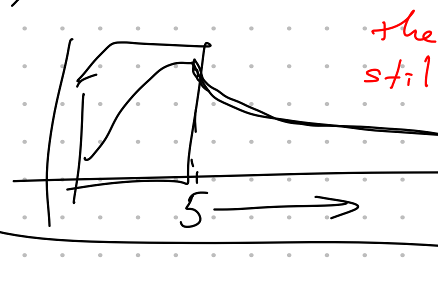
## The Integral Test

Suppose $f$ is cont., positive, decreasing on $[1,\infty)$ and let $a_n = f(n)$. Then

1) If $\int_1^{\infty} f(x)\,dx$ is CONV, then $\sum_{n=1}^{\infty} a_n$ is CONV

2) // // // DIV, // // is DIV

the test still works

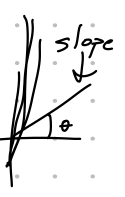



**Example** Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for CONV or DIV.

$f(x) = \frac{1}{x^2+1}$ cont., positive, decreasing for $x \geq 1$

Thus, we can apply the integral test.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_1^t$$




$$= \lim_{t\to\infty} \arctan t - \underbrace{\arctan(1)}_{\pi/4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

In particular, $\int_1^{\infty} f(x)\,dx$ is CONV and thus, $\sum_{n=1}^{\infty} a_n$ is CONV.

**Example** For what values of $p$ is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONV?

If $p \leq 0$ then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$. Thus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is DIV by the test for divergence.

If $p > 0$, then $f(x) = \frac{1}{x^p}$ is cont, positive and decreasing (for $x > 0$)

So we can use the integral test.

$\int_1^{\infty} \frac{1}{x^p}\,dx$ is CONV if and only if $p > 1$

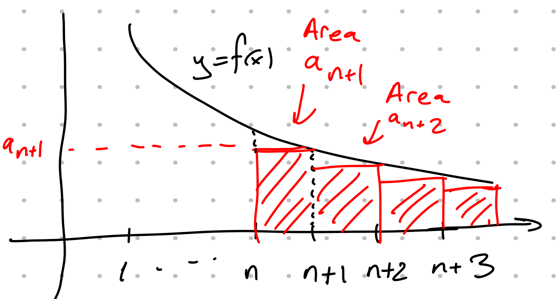
Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is CONV if and only if $p > 1$

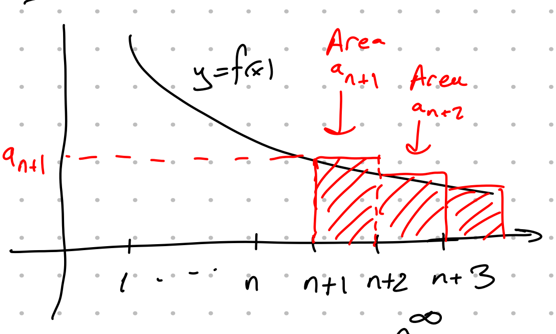
p-series

## Estimating the Sum of a Series

Given a CONV series $\sum_{n=1}^{\infty} a_n$, we want to estimate its sum $s$. We can use a partial sum $s_n = a_1 + a_2 + \cdots + a_n$ to estimate $s$ but what is the error?

Remainder $= R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$




$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} f(x)\,dx$ $\qquad$ $R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)\,dx$

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

($f$ is cont., decreasing and positive $a_n = f(n)$)

Remainder Estimate for the Integral Test

**Example** a) Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms.

b) Estimate the error involved in this approximation.

c) How many terms are required to ensure that the sum is accurate to within 0.0005?

(a) $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \overset{\text{calculator}}{\approx} 1.1975$

(b) $R_{10} = ?$ $\quad \int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_n^{t} x^{-3}\,dx = \lim_{t\to\infty} -\frac{x^{-2}}{2}\Big|_n^t$

$$= \lim_{t\to\infty} -\frac{1}{2t^2} + \frac{1}{2n^2} = \frac{1}{2n^2}$$

Thus $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{200} = 0.005$

(c) We want $n$ such that $R_n \le 0.0005$

$$R_n \le \int_n^\infty \frac{1}{x^3}dx = \frac{1}{2n^2} \le 0.0005$$

$a \ge b \to -a \le -b$

$\frac{1}{a} \le \frac{1}{b}$

$$\frac{1}{n^2} \le 0.001 \to n^2 \ge \frac{1}{0.001} = 1000$$

$$n \ge \sqrt{1000} \approx 31.6 \qquad n \ge 32$$

$$\to \int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx$$

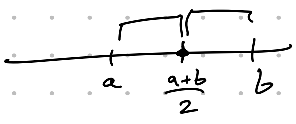
$$\boxed{S_n + \underset{S - S_n}{\underset{\parallel}{R_n}} = S}$$

$$S_n + \int_{n+1}^\infty f(x)dx \le S \le S_n + \int_n^\infty f(x)dx$$

$n=10$ $\quad f(x) = \frac{1}{x^3}$ $\quad \int_{11}^\infty \frac{1}{x^3}dx = \frac{1}{2(11)^2}$ $\quad \int_{10}^\infty \frac{1}{x^3}dx = \frac{1}{2(10)^2} = \frac{1}{200}$

$S_{10} \approx 1.197532$

$$\underset{a}{1.201664} \le S \le \underset{b}{1.202532}$$




So $S \approx 1.2021$ and error $< 0.0005$

## 11.4 The Comparison Tests

**The Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $a_n \le b_n$

i) If $\sum b_n$ is CONV then $\sum a_n$ is CONV

ii) If $\sum a_n$ is DIV then $\sum b_n$ is DIV

**Example** Determine whether the series $\sum_{n=1}^\infty \frac{5}{2n^2+4n+3}$ CONV or DIV.

$$\frac{5}{2n^2 + \underbrace{4n+3}_{\text{positive}}} \le \frac{5}{2n^2} \qquad \sum_{n=1}^\infty \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^\infty \frac{1}{n^2} \text{ CONV}$$

(p-series with $p>1$)

By the comparison test, $\sum_{n=1}^\infty \frac{5}{2n^2+4n+3}$ CONV.

**Example** Test $\sum_{k=1}^\infty \frac{\ln k}{k}$ for CONV or DIV.

$\ln k \ge 1$ only for $k \ge 3$ $\qquad \frac{\ln k}{k} \ge \frac{1}{k}$ for $k \ge 3$

Since $\sum_{k=1}^\infty \frac{1}{k}$ is DIV, $\sum_{k=1}^\infty \frac{\ln k}{k}$ is also DIV by the comparison test.

(p-series with $p \le 1$)

We can use the comparison test to show $\sum_{n=1}^\infty \frac{1}{2^n+1}$ is CONV by comparing $\frac{1}{2^n+1} \le \frac{1}{2^n}$ $\qquad \sum_{n=1}^\infty \frac{1}{2^n}$ is geometric series with $|r| = \frac{1}{2} < 1$ CONV.

However, it is harder to use the comparison test for $\sum_{n=1}^\infty \frac{1}{2^n-1}$

**The Limit Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ is finite and $c > 0$

then either both $\sum a_n$ and $\sum b_n$ CONV or both DIV.

**Example** Test $\sum_{n=1}^\infty \frac{1}{2^n-1}$ $\qquad \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{\frac{1}{\cancel{2^n}(1-\underbrace{1/2^n}_{\to 0})}}{\frac{1}{\cancel{2^n}}} = \frac{\frac{1}{1-0}}{1} = 1$$

Since $\sum \frac{1}{2^n}$ is CONV (geometric), $\sum \frac{1}{2^n-1}$ is also CONV.