Recall: Given $\{a_n\}$, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ (if the limit exists)

where $s_n = \sum_{i=1}^{n} a_i$

Example (Geometric Series)

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

r: common ratio $\qquad \left(= \sum_{n=0}^{\infty} ar^n\right)$

If $r = 1$, $s_n = \underbrace{a + a + a + a + \dots + a}_{n} = na$

Since $a \neq 0$, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} na = \pm\infty$ DIV!

If $r \neq 1$, $s_n = a + ar + ar^2 + \dots + ar^{n-1}$

$r s_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$

$s_n - r s_n = a - ar^n$

$s_n(1-r) = a(1-r^n) \quad \longrightarrow \quad s_n = \frac{a(1-r^n)}{1-r}$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$ is CONV if r^n is CONV, is DIV if r^n is DIV

Recall: If $-1 < r < 1$ then $r^n \to 0$ (as $n \to \infty$)
If $r \leq -1$ or $r > 1$ then r^n is DIV.

So If $-1 < r < 1$ (or $|r| < 1$) then $\lim_{n\to\infty} s_n = \frac{a(1-0)}{1-r} = \boxed{\frac{a}{1-r} = \sum_{n=1}^{\infty} ar^{n-1}}$

Otherwise ($|r| \geq 1$) $\sum_{n=1}^{\infty} ar^{n-1}$ is DIV.

Example Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$$

$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$ if $|r| < 1$

$a = 5$

$ar = -\frac{10}{3} \quad \rightarrow \quad \cancel{5}r = -\frac{\cancel{10}^{\,2}}{3} \qquad r = -\frac{2}{3} \qquad |r| < 1$

The sum is CONV and equal to $\frac{a}{1-r} = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{1+\frac{2}{3}} = \frac{5}{\frac{5}{3}} = 3$

Example Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ CONV or DIV?

$$= \sum_{n=1}^{\infty} (2^2)^n\, 3 \cdot 3^{-n} = \sum_{n=1}^{\infty} 3\,\frac{4^n}{3^n} = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n$$

$r = \frac{4}{3} > 1$ DIV!

Example Write the number $2.3\overline{17} = 2.3171717\dots$ as a ratio of integers.

$$2.3\overline{17} = 2.3 + \boxed{\underset{\frac{17}{10^3}}{0.017} + \underset{\frac{17}{10^5}}{0.00017} + \frac{17}{10^7} + \dots}$$

a geometric series

$a + ar + ar^2 + \dots$

$a = \frac{17}{10^3}$

$ar = \frac{17}{10^5} \quad \rightarrow \quad \cancel{\frac{17}{10^3}}\, r = \frac{\cancel{17}}{\cancel{10^5}\,10^2} \qquad r = \frac{1}{10^2} \qquad (|r| < 1 \text{ CONV})$

$$\frac{a}{1-r} = \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = \frac{17}{10^3} \cdot \frac{1}{\frac{100-1}{10^2}} = \frac{17}{10^{\cancel{3}}} \cdot \frac{\cancel{10^2}}{99} = \frac{17}{990}$$

So $2.3\overline{17} = 2.3 + \frac{17}{990} = \frac{23}{10} + \frac{17}{990} = \frac{23(99) + 17}{990} = \frac{1147}{495}$

Example Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{a}{1-r} = \frac{1}{1-x} \qquad (|x| < 1)$$

$(a + ar + ar^2 + \dots)$

$a = 1$
$r = x \qquad |r| < 1$ CONV

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad (|x| < 1)$$

Example (telescoping sum)

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is CONV and find its sum.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$$

Actually $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1} \qquad \left(\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}\right)$

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\dots} \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$s_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \cancel{\frac{1}{n+1}}^{\,0} = 1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Example Show that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

is DIV!

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right)$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$

$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right)$

similarly, $S_{16} > 1 + 4\left(\frac{1}{2}\right)$

In general $S_{2^n} > 1 + n\left(\frac{1}{2}\right) \to \infty$ as $n \to \infty$

So the Partial sums are also $\to \infty$ (DIV).

Theorem If the series $\sum_{n=1}^{\infty} a_n$ is CONV then $\lim_{n\to\infty} a_n = 0$

Converse is not true! Just because $a_n \to 0$ does not mean $\sum_{n=1}^{\infty} a_n$ is CONV

an example is the harmonic series $\sum \frac{1}{n}$

Theorem If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$

then $\sum_{n=1}^{\infty} a_n$ is DIV. ("Test for Divergence")

Example Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$a_n = \frac{n^2}{5n^2+4}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5 + \underbrace{\frac{4}{n^2}}_{0}} = \frac{1}{5} \neq 0$

by the test for divergence $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ DIV

Theorem If $\sum a_n$ and $\sum b_n$ CONV and c is a constant

then $\sum_{n=1}^{\infty} c\,a_n = c \sum_{n=1}^{\infty} a_n$

$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

(In particular these are CONV series)

Example Find $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

$= 3 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+1)}}_{=1} + \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$a = \frac{1}{2}$ $\quad r = \frac{1}{2}$

$\frac{a}{1-r} = a\frac{1}{1-r} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\frac{1}{2}} = 1$

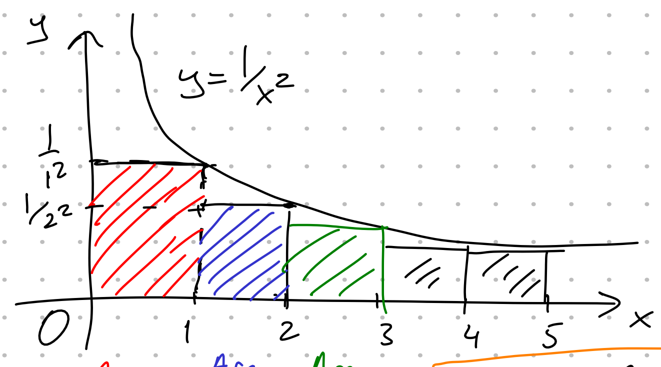
$= 3 + 1 = 4$

11.3 The Integral Test and Estimates of Sums

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Q1 Is it CONV?

Q2 If so, what is the sum?



Total Area of Boxes $= \sum_{n=1}^{\infty} \frac{1}{n^2}$

Area $= \frac{1}{1^2}$ $\quad$ Area $= \frac{1}{2^2}$ $\quad$ Area $= \frac{1}{3^2}$

(Except for the first box of area $=1$.) Total Area of Boxes $< \int_1^{\infty} \frac{1}{x^2}\,dx$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$

Note that $S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ is an increasing (monotonic) and bounded seq.

So by the Monotonic Sequence Theorem S_n is CONV.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is CONV.