Example Find $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot n}{\frac{1}{n}(n+1)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$, where $\frac{1}{n} \to 0$

$= \frac{1}{1+0} = 1$

Example Calculate $\lim_{n\to\infty} \frac{\ln(n)}{n}$

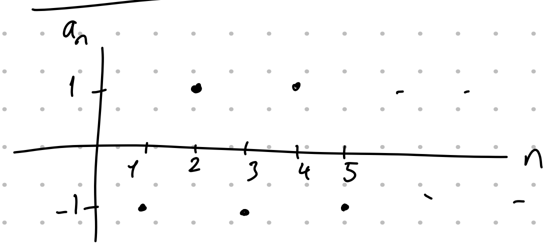
Recall: $f(x) = \frac{\ln x}{x}$ $f(n) = a_n$ and $f(x) \to L$ as $x \to \infty$

then $a_n \to L$ as $n \to \infty$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$ therefore

$\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

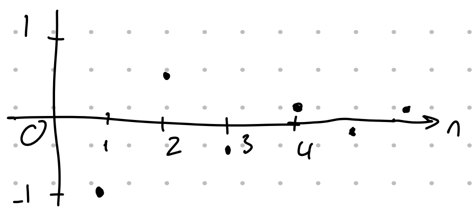
Example Determine whether $a_n = (-1)^n$ is CONV or DIV.



DIV because

Values are not approaching to any constant.

Example Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.



Set $a_n = \frac{(-1)^n}{n}$ $|a_n| = \frac{1}{n} \to 0$ as $n\to\infty$

Recall: If $|a_n| \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.

Thus, $\lim_{n\to\infty} \frac{(-1)^n}{n}$

Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L then

$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Example Find $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$.

Since $\sin$ is continuous everywhere, $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$, where $\frac{\pi}{n} \to 0$

$= \sin(0) = 0$

Example Find $\lim_{n\to\infty} \frac{n!}{n^n}$ if it exists.

$a_1 = \frac{1}{1}$ $a_2 = \frac{2!}{2^2} = \frac{1\cdot 2}{2\cdot 2}$ $a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$ . . .

$a_n = \frac{1\cdot 2\cdot 3 \cdot \dots \cdot n}{n\cdot n\cdot n \cdot \dots \cdot n} = \frac{1}{n}\underbrace{\left(\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdots\frac{n}{n}\right)}_{<1}$ (each factor ≤ 1)

So $0 < a_n \le \frac{1}{n} \to 0$ as $n \to \infty$

$0 \to 0$ as $n\to\infty$. By the Squeeze theorem, $a_n \to 0$ as $n \to \infty$

Example For what values of r is the sequence $\{r^n\}$ CONV?

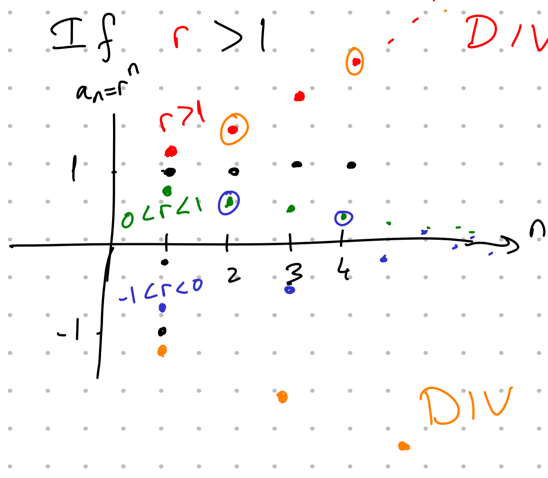
(Recall for $r=-1$, $(-1)^n$ is DIV)

$r=1$ $a_n = 1^n = 1$ constant seq. so it is conv.

$a_n \to 1$ as $n \to \infty$

$r=0$ $a_n = 0^n = 0$ so again $a_n \to 0$ as $n \to \infty$

If $r > 1$ DIV



If $0 < r < 1$ CONV

$r^n \to 0$ as $n\to\infty$

If $r < -1$ DIV

DIV

If $-1 < r < 0$

$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$

So $r^n \to 0$ as $n \to \infty$

Result:

$\{r^n\}$ is CONV if $-1 < r \le 1$ and DIV for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$$

Def^n: A seq. $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \ge 1$ that is $a_1 < a_2 < a_3 < a_4 < \dots$. It is called decreasing if $a_n > a_{n+1}$ for all $n \ge 1$

A sequence is monotonic if it is either increasing or decreasing

Example $\left\{\frac{3}{n+5}\right\}$ $a_1 = \frac{3}{1+5} = \frac{3}{6}$ $a_2 = \frac{3}{7}$ $a_3 = \frac{3}{8}$

$a_n = \frac{3}{n+5} > \frac{3}{n+6} = \frac{3}{(n+1)+5} = a_{n+1}$

$0 \le a_n \le a_1 = \frac{3}{6} = \frac{1}{2}$ $\underbrace{0 \le a_n \le 1/2}_{\text{bounded sequences}}$

Monotonic Sequence Theorem

Every bounded, monotonic sequence is CONV.

e.g. $\left\{ \frac{3}{n+5} \right\}$ is decreasing so monotonic.

It is also bounded so it must be CONV by the theorem above!

(Inc. if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$ $a_1 \le a_n$
If Inc. then it is automatically bounded from below.)

(Dec If $a_1 > a_2 > \cdots > a_n > \cdots$ $a_1 \ge a_n$
If DEC. then it is automatically bounded from above.)

11.2 Series

$\pi = 3.14159 \cdots$

$$\pi = \underset{a_1}{\underset{\shortparallel}{3}} + \underset{a_2}{\underset{\shortparallel}{\frac{1}{10}}} + \underset{a_3}{\underset{\shortparallel}{\frac{4}{100}}} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \cdots$$

Given a sequence $\{a_n\}_{n=1}^{\infty}$ we can form the expression

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ which is called

an (infinite) series. Notation:

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \sum a_n$$

Q) What is $\sum_{n=1}^{\infty} a_n$? Does it even make sense?

e.g.
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots = 1$$

$$\frac{1}{2}, \quad \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \quad \frac{15}{16}, \quad \frac{31}{32}, \cdots, 1 - \frac{1}{2^n}$$

$\frac{1}{2^n} \to 0$

$1 - 0 = 1$

these are called partial sums $\{s_n\}$

they form a sequence!

(seq $\{a_n\}$ $\rightarrow$ series $\sum_{n=1} a_n$ $\longrightarrow$ seq $\{s_n\}$ (partial sums))

$s_1 = a_1 \qquad s_2 = a_1 + a_2 \qquad s_3 = a_1 + a_2 + a_3 \quad \cdots$

$$s_n = \sum_{i=1}^{n} a_i$$

Defn Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is CONV, and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called CONV.

and we write $a_1 + a_2 + a_3 + \cdots = s$ or $\sum_{n=1}^{\infty} a_n = s$

s is called the sum. If $\{s_n\}$ is DIV. then the series is called DIV.

Notice that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \Big(\overset{= s_n}{\sum_{i=1}^{n} a_i} \Big)$ if the limit exists.

Example Suppose

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

then $\sum_{n=1}^{\infty} a_n = \boxed{\lim_{n\to\infty} s_n} = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3 + \underbrace{\frac{5}{n}}_{\to 0}} = \frac{2}{3}$

So $\sum a_n$ is CONV.