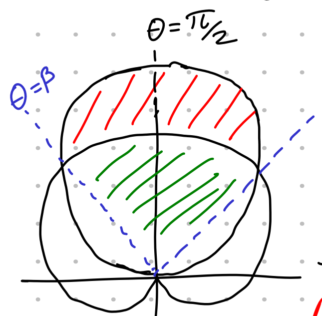
**Example** Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1+\sin\theta$




$\boxed{\text{Recall } A = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta}$

So

$$A + B = \int_\alpha^\beta \frac{1}{2}(3\sin\theta)^2 d\theta \qquad B = \int_\alpha^\beta \frac{1}{2}(1+\sin\theta)^2 d\theta$$

$3\sin\theta = 1+\sin\theta \Rightarrow 2\sin\theta = 1$

$\sin\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6} = \alpha$

$\beta = \frac{5\pi}{6}$

$$A = \frac{1}{2}\int_\alpha^\beta (3\sin\theta)^2 d\theta - \frac{1}{2}\int_\alpha^\beta (1+\sin\theta)^2 d\theta$$

$$= 2\left(\frac{1}{2}\int_\alpha^{\pi/2} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_\alpha^{\pi/2} (1+\sin\theta)^2 d\theta\right) = \int_{\alpha=\frac{\pi}{6}}^{\pi/2} \left(9\sin^2\theta - (1+2\sin\theta+\sin^2\theta)\right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\pi/2} \left[4(1-\cos 2\theta) - 1 - 2\sin\theta\right] d\theta \qquad \boxed{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)}$$

$$= \int_{\frac{\pi}{6}}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta = 3\theta - \frac{4\sin 2\theta}{2} + 2\cos\theta \Big|_{\pi/6}^{\pi/2}$$

$$= \frac{3\pi}{2} - 2\underbrace{\cancel{\sin\pi}}_{0} + 2\underbrace{\cancel{\cos\frac{\pi}{2}}}_{0} - \left(\frac{3\pi}{6} - 2\underbrace{\cancel{\sin\frac{\pi}{3}}}_{\frac{\sqrt{3}}{2}} + 2\underbrace{\cancel{\cos\frac{\pi}{6}}}_{\frac{\sqrt{3}}{2}}\right) = \frac{3\pi}{2} - \frac{\cancel{3}\pi}{\cancel{6}2}$$

$$= \pi.$$

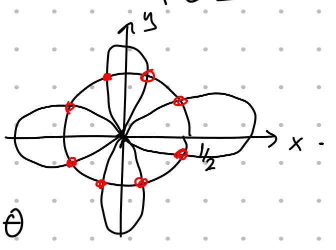
**Recall:** In polar coordinates, any point have infinitely many representatives. e.g. $(2, \frac{\pi}{6})$, $(-2, \frac{\pi}{6}+\pi)$, $(2, \frac{\pi}{6}+2\pi)$ ....

are all the same point.

**Example** Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

**Exercise:** Find 4 points that satisfy $\cos 2\theta = \frac{1}{2}$ (for $0 \le \theta \le 2\pi$) $[0, 2\pi]$

**Exercise:** Find 4 more points that satisfy $\cos 2\theta = -\frac{1}{2}$ (for $0 \le \theta \le 2\pi$)




## Arc Length

$r = f(\theta) \quad a \le \theta \le b$

$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \qquad \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= (f'(\theta))^2\cos^2\theta - \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta} + (f(\theta))^2\sin^2\theta$$

$$+ (f'(\theta))^2\sin^2\theta + \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta} + (f(\theta))^2\cos^2\theta$$

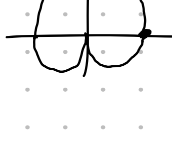
$$= (f'(\theta))^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + (f(\theta))^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = (f'(\theta))^2 + (f(\theta))^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

thus

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \qquad (r = f(\theta))$$

**Example** Setup the integral to find the length of the cardioid $r = 1+\sin\theta \quad 0 \le \theta \le 2\pi$



$$L = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \int_0^{2\pi} \sqrt{\cos^2\theta + (1+\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

# 11 Infinite Sequences and Series

## 11.1 Sequences

$a_1, a_2, a_3, \ldots, a_n, \ldots$

the first term, the second term, the $n^{th}$ term, the general term

**Notations:** $\{a_1, a_2, a_3, \ldots\}$ $\quad \{a_n\}$ or $\{a_n\}_{n=1}^\infty$

**Examples**

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^\infty \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$

b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\} \qquad a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left\{-\frac{2}{3}, \frac{3}{3^2} = \frac{1}{3}, \frac{-4}{3^3}, \ldots\right\}$

c) $\left\{\sqrt{n-3}\right\}_{n=3}^\infty \qquad a_n = \sqrt{n-3} \qquad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots\right\}$

**Example** Find a formula for the general term $a_n$ of the sequence

$\left\{ \frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125}, \dots \right\}$

$a_1 = \frac{3}{5}$, $a_2 = \frac{-4}{25}$, $a_3 = \frac{5}{125}$, — — — .

$$a_n = (-1)^{n-1}\frac{(n+2)}{5^n}$$

Example (The Fibonacci Sequence)

$\{f_n\}$ $\quad f_1 = 1, \; f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

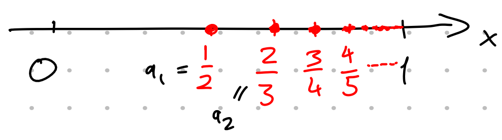
$f_4 = f_3 + f_2 = 2 + 1 = 3$

$1, 1, 2, 3, 5, 8, 13, \dots$

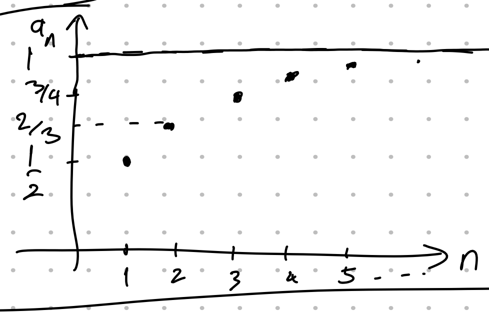
Plotting Sequences

$a_n = \frac{n}{n+1} \qquad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

1st Way



2nd Way

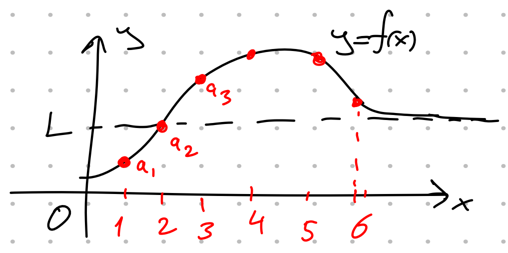


$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general: A sequence has the Limit $L$ and we write $\left(\lim_{n\to\infty} a_n = L\right)$ or $\left(a_n \to L \text{ as } n \to \infty\right)$ if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges.

Otherwise, diverges.

Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when $n$ is an integer then $\lim_{n\to\infty} a_n = L$

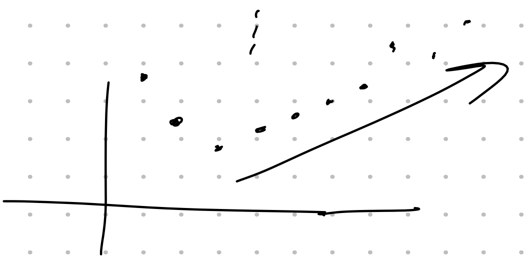


Def$^n$ $\lim_{n\to\infty} a_n = \infty$ means that for every positive number $M$ there is an integer $N$ such that if $n > N$ then $a_n > M$

e.g. $a_n = 2n + 50$

Given $M = 70$, for all $n > N = 10$, we have $a_n > M = 70$

Given $M = 250$, for all $n > N = 100$, we have $a_n > M = 250$

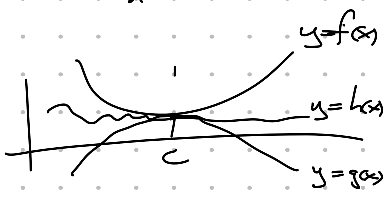


Limit Laws for Sequences $\quad \{a_n\}$ and $\{b_n\}$ are convergent and $c$ is a constant.

then

- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c\,a_n = c\left(\lim_{n\to\infty} a_n\right)$ $\qquad$ • $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ if $p > 0$ and $a_n > 0$

The Sequeeze Theorem for sequences



If $a_n \leq b_n \leq c_n$ for $n \geq n_0$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$.

Theorem If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

$-|a_n| \leq a_n \leq |a_n|$ $\quad$ so this follows from the Squeeze theorem.

$\downarrow \qquad\qquad \downarrow$

$0 \qquad\qquad 0$