The Second Midterm is Next Week (Thursday 8pm / Friday 8am)

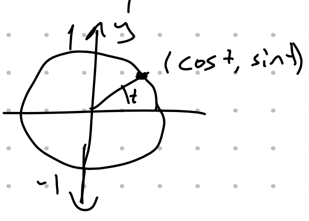
Sign up link will be sent today!

7.4 (Partial Fractions) – 10.2 Calculus with Parametric curves

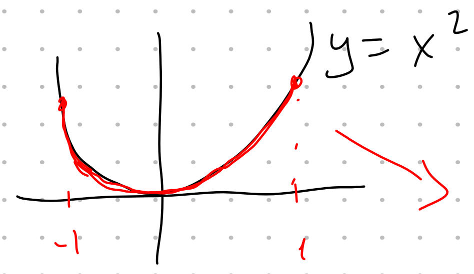
---

Example Sketch the curve with parametric equations

$x = \sin t \quad y = \sin^2 t.$



$y = \sin^2 t = (\sin t)^2 = x^2$



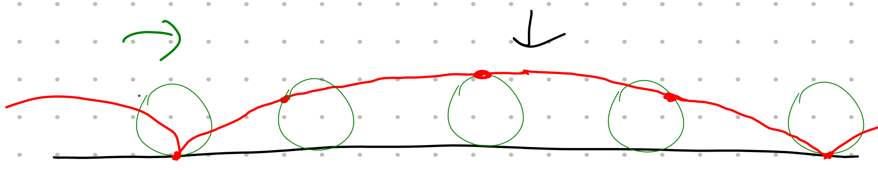
Notice that

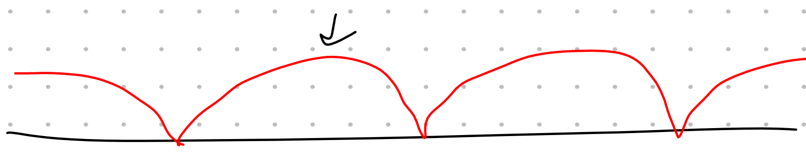
$-1 \le x = \sin t \le 1$

$y = x^2$ and $-1 \le x \le 1$

Example (The Cycloid)

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.





$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$

r: radius of the circle the parameter

10.2 Calculus with Parametric Curves

$x = f(t) \qquad y = g(t) \qquad$ what is $\dfrac{dy}{dx} = ?$

Say h is a function of x which in turn is a function of t. $h(x(t))$

By the chain rule

$$\frac{dh}{dt} = \frac{dh}{dx}\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dh}{dx} \overset{(*)}{=} \frac{\frac{dh}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

If we take $h = y$ in $(*)$, we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\text{if } \frac{dx}{dt} \neq 0\right)$$

So 1) if $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$ then the tangent line is horizontal

2) if $\dfrac{dy}{dt} \neq 0$ and $\dfrac{dx}{dt} = 0$ then the tangent line is vertical

Q| What about concavity, $\dfrac{d^2y}{dx^2}$?

In $\dfrac{dh}{dx} \overset{(*)}{=} \dfrac{\frac{dh}{dt}}{\frac{dx}{dt}}$, take $h = \dfrac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0$$

Example A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$

a) Show that C has two tangents at $(3,0)$ and find their equations.

b) Find the points on C where the tangent is horizontal or vertical.

c) Determine where the curve is concave up / down.

d) Sketch the curve.

ⓐ Set $(x,y)=(3,0)$

$x=t^2=3 \qquad y=t^3-3t=0$

$t=\pm\sqrt{3} \qquad y=t(t^2-3)=0$

$(t=0, \pm\sqrt{3})$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t^2-3}{2t}\Bigg|_{t=\pm\sqrt{3}}$$

$$=\frac{3(3)-3}{\pm 2\sqrt{3}}=\pm\frac{6}{2\sqrt{3}}=\pm\frac{3}{\sqrt{3}}=\pm\sqrt{3}$$

$y-y_0=m(x-x_0)$

$y-0=\sqrt{3}(x-3)$ and $y-0=-\sqrt{3}(x-3)$

$y=\sqrt{3}x-3\sqrt{3}$ and $y=-\sqrt{3}x+3\sqrt{3}$

$(t=\sqrt{3}) \qquad (t=-\sqrt{3})$

ⓑ $\dfrac{dy}{dx}=\dfrac{3t^2-3}{2t}$ $\qquad 3t^2-3=0$, $3t^2=3$, $t^2=1$, $t=\pm 1$

$2t=0 \Rightarrow t=0$

$3t^2-3=0 \Rightarrow t=\pm 1$ and at $t=\pm 1$ $2t\neq 0$

$t=\pm 1$ gives us horizontal tangents.

$2t=0 \Rightarrow t=0$ and at $t=0$ $3t^2-3\neq 0$

$t=0$ gives us the only vertical tangent

$t=\pm 1 \Rightarrow (x,y)=(t^2, t^3-3t)=(1, \mp 2) \leftarrow$ hor. tangt

$t=0 \Rightarrow (x,y)=(0,0) \leftarrow$ vert. tangent

ⓒ
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

$$=\frac{\frac{6t(2t)-(3t^2-3)2}{(2t)^2}}{2t}=\frac{12t^2-6t^2+6}{(2t)^3}$$

$$=\frac{6t^2+6}{8t^3}$$

numerator is always positive

$8t^3>0$ if $t>0$

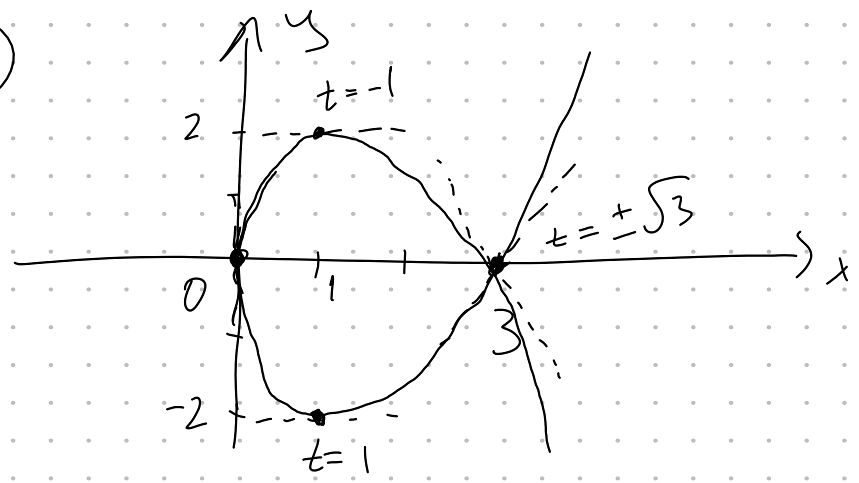
$8t^3<0$ if $t<0$

So $\dfrac{d^2y}{dx^2}$ is positive if $t>0$

$\dfrac{d^2y}{dx^2}$ is negative if $t<0$

In other words, the curve is concave up for $t>0$ and it is concave down for $t<0$.

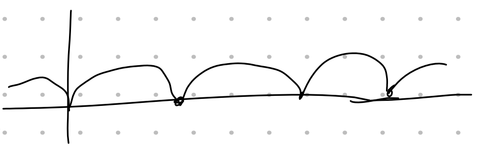
ⓓ



Example a) Find the tangent to the cycloid

$x=r(\theta-\sin\theta)$, $y=r(1-\cos\theta)$ at the point where $\theta=\pi/3$

b) At what points is the tangent horizontal? When is it vertical?



$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\cancel{r}\sin\theta}{\cancel{r}(1-\cos\theta)}\Bigg|_{\theta=\pi/3}$$

$$\frac{\sin(\pi/3)}{1-\cos(\frac{\pi}{3})} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}$$

$m = \sqrt{3}$. $\quad x_0 = r(\theta - \sin\theta) = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$y_0 = r(1-\cos\theta) = r\left(1-\cos\frac{\pi}{3}\right) = r\left(1-\frac{1}{2}\right) = \frac{r}{2}$

$y - y_0 = m(x - x_0)$

$y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$

(b) $\frac{dy}{dx} = \frac{\sin\theta}{(1-\cos\theta)}$ horizontal tangent when $\sin\theta = 0$ but $1-\cos\theta \neq 0$

$\sin\theta = 0 \Rightarrow \theta = ..., -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, ...$

$\theta = k\pi$ (integer multiples of π)

$1-\cos\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow \theta = ..., -2\pi, 0, 2\pi, 4\pi...$

$\theta = 2n\pi$ (even multiples of π)

So for odd multiples of π, we have

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta} = 0}{\frac{dx}{d\theta} \neq 0}$ $\quad \theta = (2k-1)\pi$ is where the horizontal tangent lines are.

Vertical

$\frac{dy}{dx} = \frac{\sin\theta \neq 0}{1-\cos\theta = 0} \rightarrow \theta = 2n\theta$

for $\theta = 2n\theta$ $\sin\theta$ is also 0.

(To be completed next time)