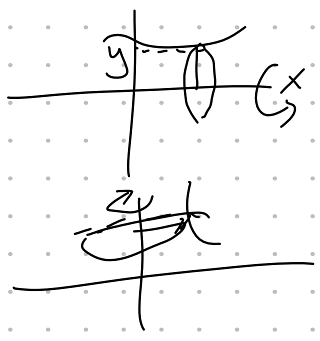
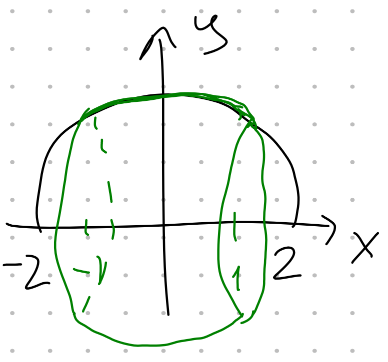
Recall: $2\pi r \ell$ $\qquad \ell = ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$S=\int 2\pi r\, ds$ $\qquad$ $r=y$ if surface is obtained by revolution about the x-axis



$r=x$ if surface is obtained by revolution about the y-axis

Example The curve $y=\sqrt{4-x^2}$, $-1\le x\le 1$ is rotated about ~~the~~ the x-axis. Find the area of the resulting surface.



$$S=\int 2\pi r\, ds = \int_{-1}^{1} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$y=\sqrt{4-x^2}=(4-x^2)^{1/2} \qquad y'=\frac{1}{\cancel{2}}(4-x^2)^{-1/2}(-\cancel{2}x)$$

$$y' = -\frac{x}{(4-x^2)^{1/2}}$$

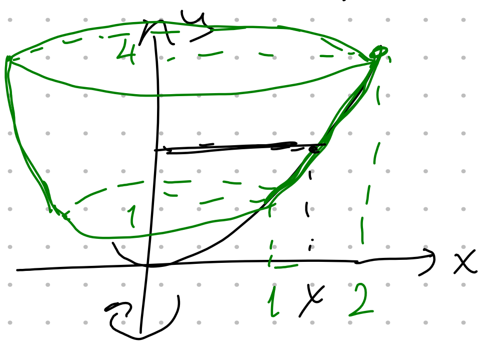
$$S=\int_{-1}^{1} 2\pi(4-x^2)^{1/2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx$$

$$=\int_{-1}^{1} 2\pi(4-x^2)^{1/2}\left(\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}\right)^{1/2}dx$$

$$=2\pi\int_{-1}^{1}\left(\cancel{(4-x^2)}\,\frac{4}{\cancel{(4-x^2)}}\right)^{1/2}dx = 2\pi\int_{-1}^{1} 2\,dx$$

$$=4\pi x\Big|_{-1}^{1} = 8\pi$$

Example $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface



SOL1 $\qquad$ ($r=x$ since the curve is rotated about the y-axis)

$$S=\int 2\pi r\, ds$$

$$=\int_{1}^{2} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$y=x^2$

$y'=2x$ $\qquad = 2\pi\int_{1}^{2} x\sqrt{1+4x^2}\,dx$

$u=1+4x^2$
$du=8x\,dx$
$x=2 \to u=1+4(2)^2=17$
$x=1 \to u=1+4=5$

$$=\frac{2\pi}{8}\int_{5}^{17} u^{1/2}\,du$$

$$=\frac{\pi}{\underset{2}{\cancel{4}}}\,\frac{\cancel{2}u^{3/2}}{3}\Bigg|_{5}^{17}$$

$$=\frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$

SOL2 $\qquad S=\int 2\pi r\, ds$ $\qquad$ ($r=x$ again!)

$$=\int_{1}^{4} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$y=x^2 \to x=y^{1/2}$ $\qquad x'=\frac{1}{2}y^{-1/2}$

$$=2\pi\int_{1}^{4} y^{1/2}\sqrt{1+\frac{1}{4y}}\,dy = 2\pi\int_{1}^{4} y^{1/2}\left(\frac{4y+1}{4y}\right)^{1/2}dy$$

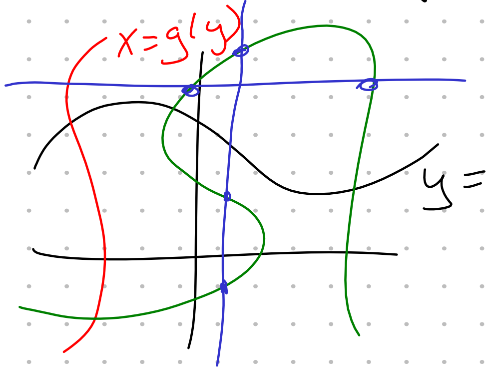
$$=\cancel{2}\pi\int_{1}^{4}\left(\cancel{y}\,\frac{4y+1}{\cancel{4y}}\right)^{1/2}dy = \pi\int_{1}^{4}(4y+1)^{1/2}\,dy$$

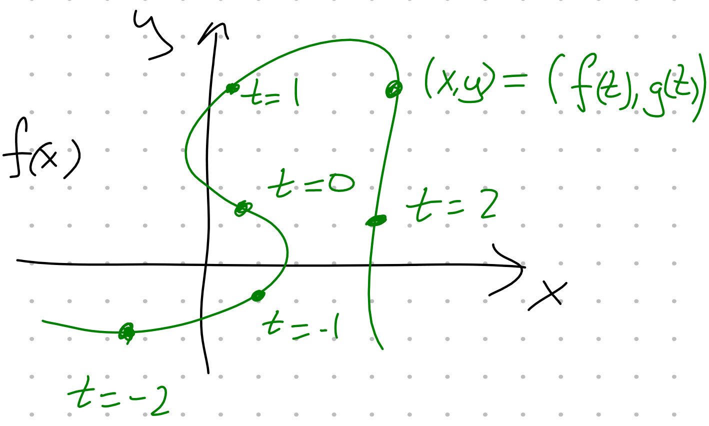
$u=4y+1 \to du=4\,dy$
$y=4 \to u=1+16=17$
$y=1 \to u=1+4=5$

$$=\frac{\pi}{4}\int_{5}^{17} u^{1/2}\,du$$

$$= \frac{\pi}{4}\left(\frac{2}{3}\right) u^{3/2} \Big|_5^{17} = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

10.1 Curves Defined by Parametric Equations





Here x and y coordinates of the particle are functions of a third variable t called the parameter and

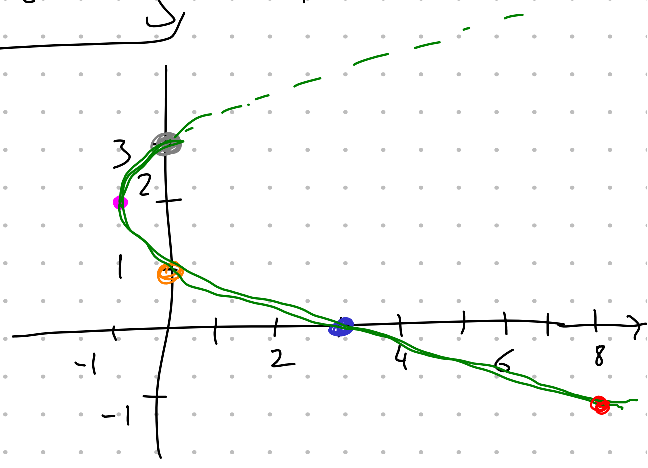
$$x = f(t) \qquad y = g(t)$$

called the parametric equations.

Example Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3



$y = t + 1 \to t = y - 1$

Thus $x = t^2 - 2t = (y-1)^2 - 2(y-1)$

$x = y^2 - 2y + 1 - 2y + 2$

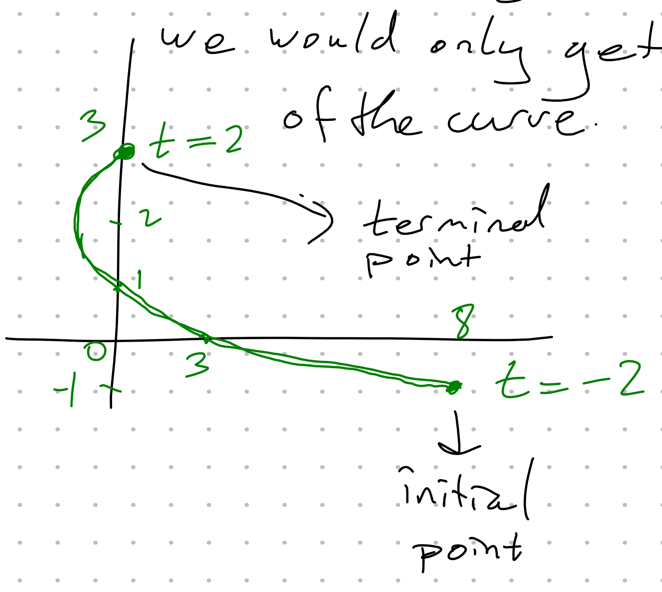
$x = y^2 - 4y + 3$ (parabola)

In the example above there was no restriction on t and that gave us the whole parabola $x = y^2 - 4y + 3$

If we had

$x = t^2 - 2t \quad y = t + 1 \qquad -2 \le t \le 2$

we would only get a portion of the curve

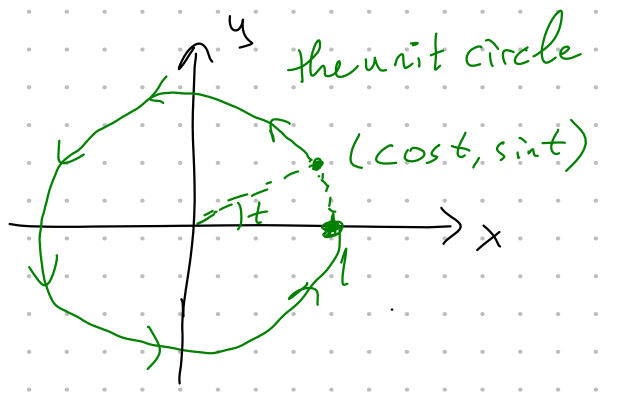


$x = y^2 - 4y + 3$

$-1 \le y \le 3$

Example What curve is represented by the following parametric equations?

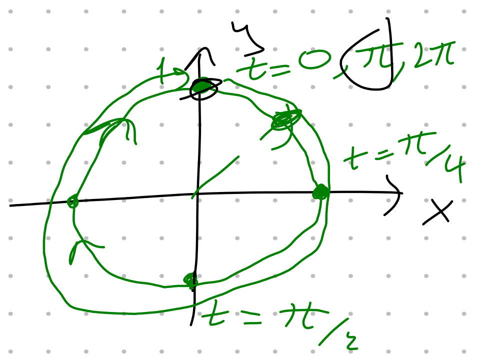
$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$



$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

Example $x = \sin 2t \qquad y = \cos 2t \qquad 0 \le t \le 2\pi$



$t = \frac{\pi}{4} \qquad x = \sin\frac{\pi}{2} = 1$

$y = \cos\frac{\pi}{2} = 0$

$x^2 + y^2 = \sin^2 2t + \cos^2 2t$

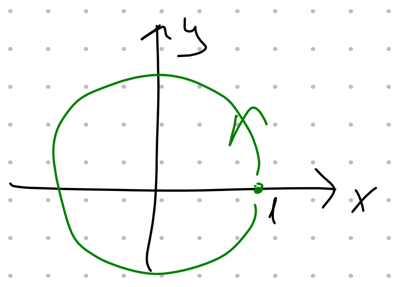
$x^2 + y^2 = 1$

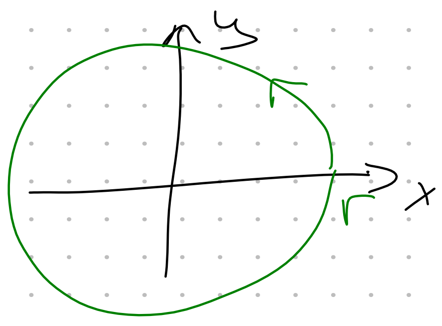
Example Find the parametric equations for the circle with center (h, k) and radius r.

$x = \cos t$
$y = \sin t$
$0 \le t \le 2\pi$

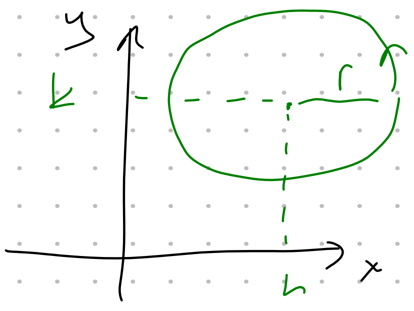
$\rightarrow$

$x = r\cos t$
$y = r\sin t$
$0 \le t \le 2\pi$





$$\boxed{\begin{aligned} x &= r\cos t + h \\ y &= r\sin t + k \\ 0 &\le t \le 2\pi \end{aligned}}$$



Example Sketch and identify

$x = \sin t, \quad y = \sin^2 t$