Comparison theorem

$f(x) \geq g(x) \geq 0 \quad x \geq a$

a) If $\int_a^\infty f(x)\,dx$ is CONV then $\int_a^\infty g(x)\,dx$ is CONV

b) If $\int_a^\infty g(x)\,dx$ is DIV then $\int_a^\infty f(x)\,dx$ is DIV

Example $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$. Is it CONV or DIV?

$1+e^{-x} > 1 \Rightarrow \frac{1+e^{-x}}{x} > \frac{1}{x}$

Since $\int_1^\infty \frac{1}{x}\,dx$ is DIV, $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is DIV

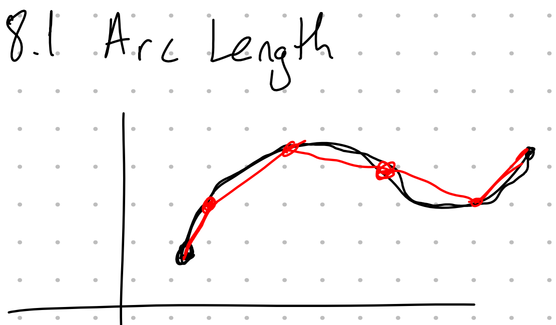
Example Is $\int_1^\infty \frac{1}{(x+1)^2}\,dx$ DIV or CONV?

$(x+1)^2 > x^2 \rightarrow \frac{1}{(x+1)^2} < \frac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ is CONV, $\int_1^\infty \frac{1}{(x+1)^2}\,dx$ is CONV.

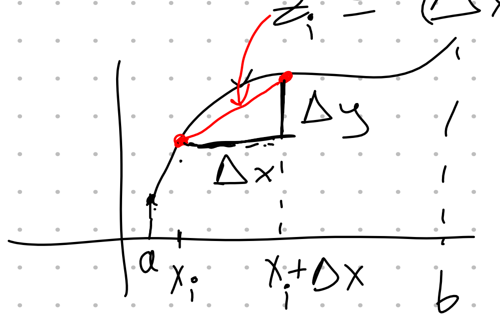
$$\boxed{\text{p-test: } \int_1^\infty \frac{1}{x^p}\,dx \quad \begin{array}{l} \text{is CONV if } p>1 \\ \text{is DIV if } p \leq 1 \end{array}}$$

8.1 Arc Length



the length of the arc $\approx$ total length of the line segments

$z_i^2 = (\Delta x)^2 + (\Delta y)^2$



$$z_i = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \sqrt{(\Delta x)^2\left(1+\left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$
$$= \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

$$\text{Arc length} = \lim_{n\to\infty} \sum_{i=1}^{n} z_i$$
$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$
$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$\frac{\Delta y}{\Delta x} \rightarrow \frac{dy}{dx} = f'(x)$

Example Find the length of the arc of $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$

$y = x^{3/2} = f(x) \qquad f'(x) = \frac{3}{2}x^{1/2}$

$$\text{Arc Length} = \int_1^4 \sqrt{1+\left(\tfrac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^4 \sqrt{1+\tfrac{9}{4}x}\,dx$$

$u = 1+\frac{9}{4}x \qquad du = \frac{9}{4}dx \rightarrow \frac{4}{9}du = dx$

$x=4 \rightarrow u = 1+\frac{9}{\cancel{4}}\cancel{4} = 10$

$x=1 \rightarrow u = 1+\frac{9}{4} = \frac{13}{4}$

$$= \int_{13/4}^{10} \frac{4}{9}u^{1/2}\,du$$

$$= \frac{4}{9}\left(\frac{2}{3}\right)u^{3/2}\Big|_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right)$$

Example Set up an integral for the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $\left(\frac{1}{4},\frac{1}{2}\right)$

$$\text{Arc length} = \int_0^{1/2} \sqrt{1+(f'(y))^2}\,dy$$

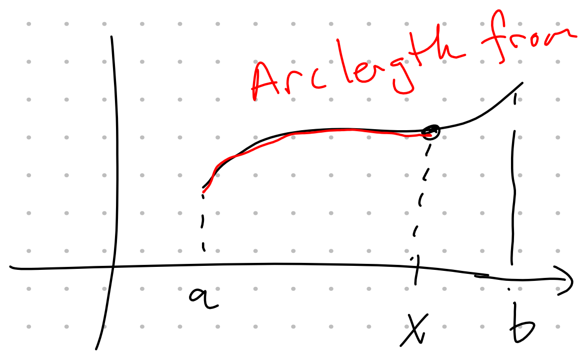
$f(y) = y^2$

$f'(y) = 2y$

$$= \int_0^{1/2} \sqrt{1+4y^2}\, dy$$

The Arc Length Function

$$\boxed{z_i = \sqrt{(\Delta x)^2 + (\Delta y)^2}}$$



$$\boxed{s(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt}$$

By FTC, $\dfrac{ds}{dx} = \sqrt{1+(f'(x))^2} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Notice that

$$\int_a^b \sqrt{1+(f'(x))^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^k ds$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{(dx)^2+(dy)^2}$$

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2+1}\, dy$$

Example Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $(1,1)$ as the starting point. (assume $x \geq 1$)

$f(x) = x^2 - \frac{1}{8}\ln x \qquad f'(x) = 2x - \frac{1}{8x}$

$$s(x) = \int_1^x \sqrt{1+(f'(t))^2}\, dt = \int_1^x \sqrt{1+\left(2t-\frac{1}{8t}\right)^2}\, dt$$

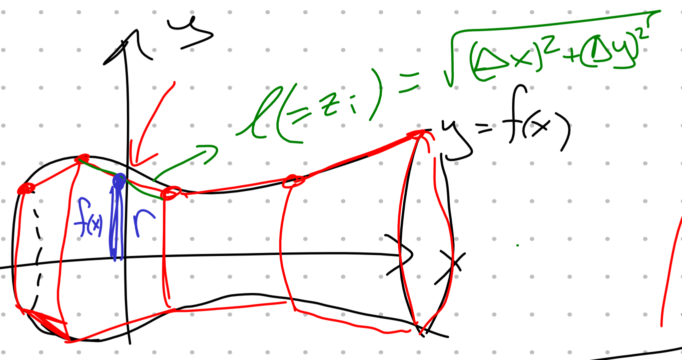
$$= \int_1^x \sqrt{1+4t^2-\frac{1}{2}+\frac{1}{64t^2}}\, dt$$

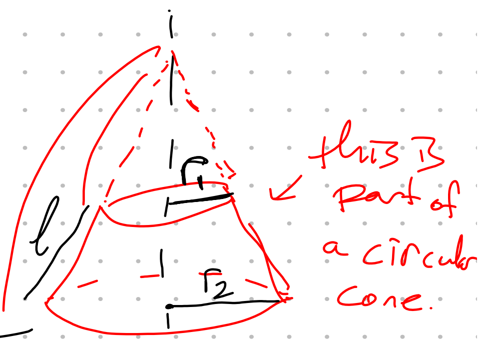
$$= \int_1^x \sqrt{4t^2+\frac{1}{2}+\frac{1}{64t^2}}\, dt = \int_1^x \sqrt{\left(2t+\frac{1}{8t}\right)^2}\, dt$$

$$= \int_1^x \left(2t+\frac{1}{8t}\right) dt = t^2 + \frac{1}{8}\ln t \Big|_1^x$$

$$s(x) = x^2 + \frac{1}{8}\ln x - 1 \qquad (\ln 1 = 0)$$

8.2 Area of a Surface of Revolution





surface area of the bottom part of cone $= 2\pi r \ell$ where $r = \dfrac{r_1+r_2}{2}$

$y = f(x) \qquad ds = \sqrt{1+(f'(x))^2}\, dx$

$\qquad\qquad f'(x) = \dfrac{dy}{dx}$

Area of the surface of revolution $S \approx \sum_{i=1}^{n} 2\pi r_i \ell_i$

$2\pi r \ell$

$r \to y$

$$\boxed{\begin{aligned} ds &= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \end{aligned}}$$

$$S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

or $S = \int_c^d 2\pi \underbrace{y}_{r} \underline{\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy}$

$2\pi r \ell$

or $S = \int 2\pi y \, \underline{ds}$

When the surface is rotated about the x-axis