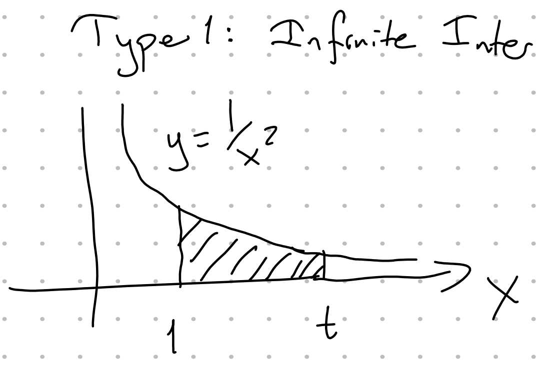
7.8 Improper Integrals

Type 1: Infinite Intervals



$$A(t) = \int_1^t \frac{1}{x^2}\,dx = \int_1^t x^{-2}\,dx$$

$$= -x^{-1}\Big|_1^t = -t^{-1} - (-1)$$

$$= -\frac{1}{t} + 1 = 1 - \frac{1}{t}.$$ Notice that $\lim_{t\to\infty} A(t) = 1$

In this case, $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$

Def^n a) If $\int_a^t f(x)\,dx$ exists for all $t \geq a$,

then $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$ provided

this limit exists (as a finite number)

b) If $\int_t^b f(x)\,dx$ exists for all $t \leq b$,

then $$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$ provided

this lim. exists.

$\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called <u>convergent</u>

if the corresponding limit exists, <u>divergent</u>

otherwise.

c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$

are convergent then we <u>define</u>

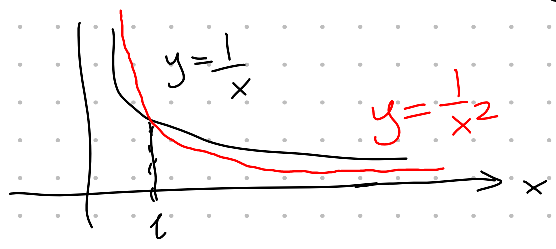
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

<u>Example</u> Determine whether the integral

$\int_1^\infty \frac{1}{x}\,dx$ is <u>convergent</u> or <u>divergent</u>.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big|_1^t$$

$$= \lim_{t\to\infty}\left(\ln t - \underbrace{\ln 1}_{=0}\right) = \infty \quad (\underline{\text{div}})$$



<u>Example</u> $$\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

$$= \lim_{t\to-\infty}\left(xe^x\Big|_t^0 - \int_t^0 e^x\,dx\right)$$

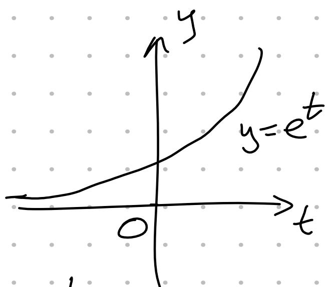
$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= \lim_{t\to-\infty}\left(0 - te^t - e^x\Big|_t^0\right)$$

$$= \lim_{t\to-\infty}\left(\underset{\downarrow}{-te^t}\ \underset{\downarrow}{-\,1 +}\ \underset{\downarrow}{e^t}\right) = -1$$

$\qquad\qquad 0 \qquad -1 \qquad 0$

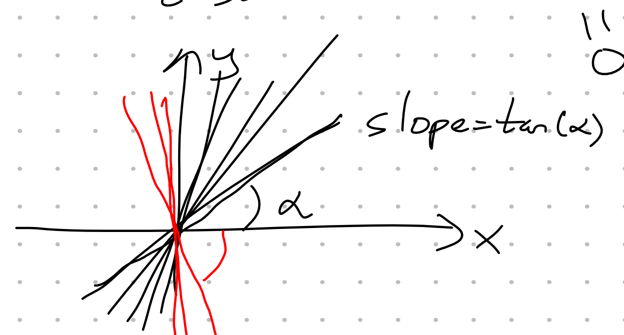


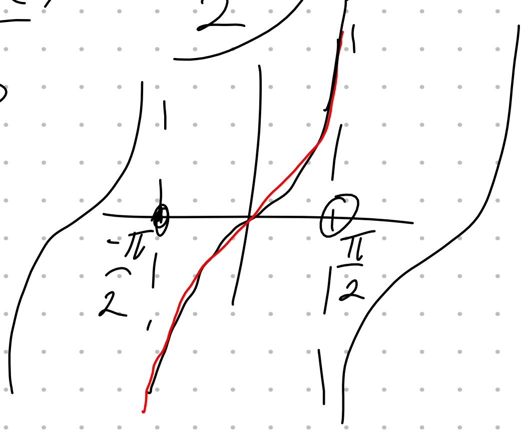
$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} \overset{L'H}{=} \lim_{t\to-\infty}\frac{1}{-e^{-t}} = 0$$

<u>Example</u> $$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1}x \Big|_0^t$$

$$= \lim_{t\to\infty} \tan^{-1}(t) - \underset{0}{\underset{\shortparallel}{\tan^{-1}(0)}} = \frac{\pi}{2}$$

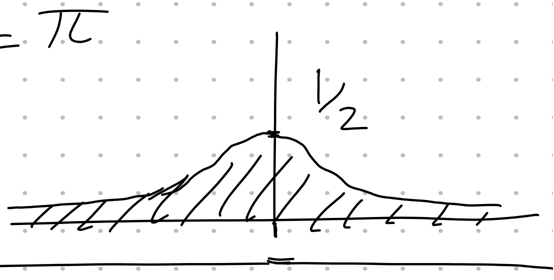




$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \tan^{-1}x \Big|_t^0$$

$$= \lim_{t\to-\infty} \underset{0}{\underset{\shortparallel}{\tan^{-1}0}} - \tan^{-1}t = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$



Example For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{convergent?}$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

If $-p+1>0$ the limit D.N.E (∞)

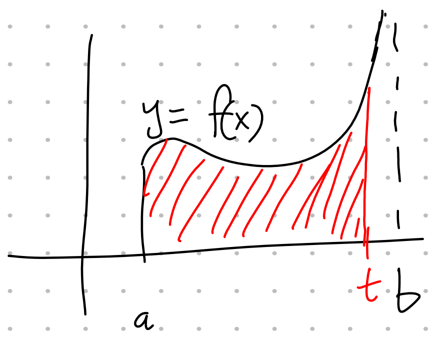
If $-p+1<0$ the limit is $\frac{-1}{-p+1}$

So $\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ is CONV if } 1<p}$

(recall if $p=1$ the integral is DIV)

$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \text{ is DIV if } p\le 1}$

Type 2: Discontinuous Integrands



$$A(t) = \int_a^t f(x)\,dx$$

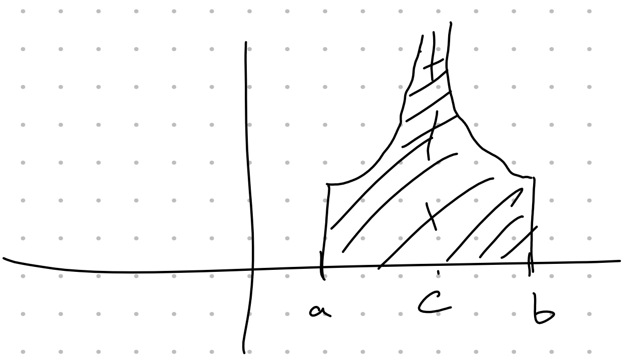
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

(if f is discontinuous at b)

If f is dis cont. at a,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

If $a<c<b$ and f is discont. at c



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

both have to be CONV.

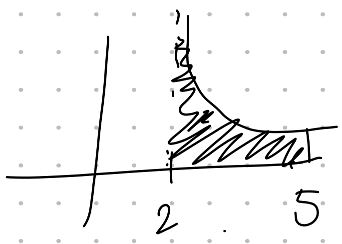
Example $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ Vert. asymp at $\underline{\underline{2}}$

$$= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 (x-2)^{-1/2}\,dx$$

$$= \lim_{t\to 2^+} 2(x-2)^{+1/2}\Big|_t^5 = \lim_{t\to 2^+} 2\sqrt{3} - \frac{2\sqrt{t-2}}{1}$$

$= 2\sqrt{3}$

CONV



Example $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ Wrong sol:

$= \ln|x-1| \Big|_0^3 = \ln|3-1| - \ln|0-1| = \ln 2 - \ln 1 = \ln 2$

Vert. Asymp at $x=1$

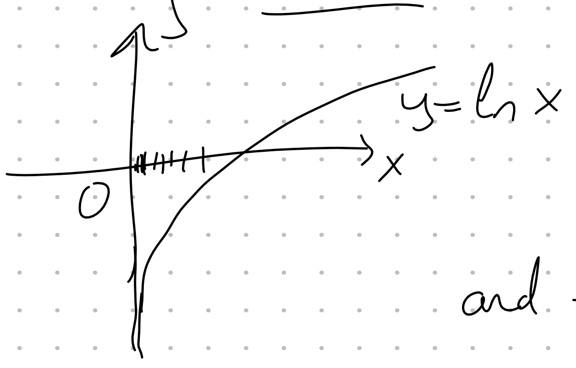
$$\int_0^3 \frac{1}{x-1}\,dx = \underbrace{\int_0^1 \frac{1}{x-1}\,dx}_{-\infty} + \underbrace{\int_1^3 \frac{1}{x-1}\,dx}_{\infty}$$

(If both are CONV)

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$$

$|t-1| = -(t-1) = -t+1 = 1-t$

$$= \lim_{t\to 1^-} \ln|x-1| \Big|_0^t = \lim_{t\to 1^-} \underbrace{\ln(\underbrace{1-t}_{0^+})}_{-\infty} - \underbrace{\ln 1}_{0}$$



So $\displaystyle\int_0^1 \frac{1}{x-1}\,dx$ is DIV

and thus $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ is DIV

Comparison theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

a) If $\displaystyle\int_a^\infty f(x)\,dx$ is CONV then $\displaystyle\int_a^\infty g(x)\,dx$ is CONV

b) If $\displaystyle\int_a^\infty g(x)\,dx$ is DIV then $\displaystyle\int_a^\infty f(x)\,dx$ is DIV