Example $\dfrac{4x}{x^3-x^2-x+1} = \dfrac{R(x)}{Q(x)}$

$Q(1)=0$ this means $x-1$ is a factor of $Q(x)$. $Q(x) = x^2(x-1)-(x-1) = (x-1)(x^2-1)$
$= (x-1)(x-1)(x+1) = (x-1)^2(x+1)^1$

$$\boxed{\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}}$$

$$\int \frac{4x}{x^3-x^2-x+1}dx = A\int\frac{1}{x-1}dx + B\int\frac{1}{(x-1)^2}dx + C\int\frac{1}{x+1}dx$$

$$= A\ln|x-1| + B\int(x-1)^{-2}dx + C\ln|x+1| + K$$

$\begin{cases} u = x-1 \\ du = dx \end{cases}$

$$= A\ln|x-1| - Bu^{-1} + C\ln|x+1| + K$$

$$= A\ln|x-1| - B(x-1)^{-1} + C\ln|x+1| + K$$

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$= (x-1)^2(x+1)$

$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$x=1$ $\quad 4 = 2B \rightarrow B=2$

$x=-1$ $\quad -4 = 4C \rightarrow C=-1$

$x=0$ $\quad 0 = -A+B+C = -A+2-1$

$A = 2-1 = 1$

$\int \frac{R(x)}{Q(x)}dx$

$$= \ln|x-1| - 2(x-1)^{-1} - \ln|x+1| + K$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated

e.g. If ax^2+bx+c is a factor of $Q(x)$ and $b^2-4ac<0$ then

$\dfrac{Ax+B}{ax^2+bx+c}$ is going to be in the decomposition.

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

$$\boxed{\int\frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

Example $\displaystyle\int\frac{2x^2-x+4}{x^3+4x}dx$ $\quad Q(x)=x^3+4x$
$= x(x^2+4)$
(the only root is 0)

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C)x$$

$$2x^2-x+4 = (A+B)x^2 + Cx + 4A$$

Two polynomials are equal if all coeff. are equal.

$2 = A+B$
$-1 = C$
$4 = 4A \rightarrow A=1$

$2 = 1+B$
$\rightarrow B=1$

$$\int\frac{2x^2-x+4}{x^3+4x}dx = \int\left(\frac{A}{x}+\frac{Bx+C}{x^2+4}\right)dx$$

$$= \int\frac{1}{x}dx + \int\frac{x-1}{x^2+4}dx$$

$$= \ln|x| + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$u = x^2+4$
$du = 2x\,dx$

(from the formula above)

$$= \ln|x| + \frac{1}{2}\int \frac{1}{u}du - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Example $\displaystyle\int \frac{5x^2+35}{3x^2-12x+15}dx$ Poly. long division first

$$\begin{array}{r} \frac{5}{3} \\ 3x^2-12x+15 \,\overline{)\,5x^2 \quad +35} \\ -\;\; 5x^2-20x+25 \\ \hline R(x) = 20x+10 \end{array}$$

$5x^2+35 = \frac{5}{3}(3x^2-12x+15) + 20x+10$

$$\frac{5x^2+35}{3x^2-12x+15} = \frac{5}{3} + \frac{20x+10}{3x^2-12x+15}$$

$$\frac{20x+10}{3x^2-12x+15} = \cancel{\frac{Ax+B}{3x^2-12x+15}}$$ (already of this form)

b^2-4ac
$12^2-4(3)(15)$
$= 12(12)-12(15) < 0$

First complete $3x^2-12x+15$ to a square.
$= 3(x^2-4x+5)$
$= 3((x-2)^2+1)$

$$\int \frac{20x+10}{3x^2-12x+15}dx = \frac{10}{3}\int \frac{2x+1}{(x-2)^2+1}dx$$

$u = x-2 \quad du = dx$
$x = u+2$

$$= \frac{10}{3}\int \frac{2(u+2)+1}{u^2+1}du$$

$$= \frac{10}{3}\int \frac{2u+5}{u^2+1}du = \frac{10}{3}\left(2\int \frac{u}{u^2+1}du + 5\int \frac{1}{u^2+1}du\right)$$

$v = u^2+1$
$dv = 2u\,du$

$$= \frac{10}{3}\left(\int \frac{1}{v}dv + 5\tan^{-1}(u)\right) + C$$

$$= \frac{10}{3}\left(\ln(u^2+1) + 5\tan^{-1}(u)\right) + C$$

$$= \frac{10}{3}\left(\ln((x-2)^2+1) + 5\tan^{-1}(x-2)\right) + C$$

$$+\frac{5}{3}x$$

Case IV: $Q(x)$ contains irreducible quadratic factors which are repeated.

e.g. $(ax^2+bx+c)^r$ is a factor $Q(x)$
$b^2-4ac<0$

then

$$\frac{A_1x+B_1}{ax^2+bx+c}, \frac{A_2x+B_2}{(ax^2+bx+c)^2}, \cdots \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Example Write down the partial fraction decomposition of

$$\frac{x^3+x^2+1}{x^4(x-1)(x^2+x+1)(x^2+1)^3} \quad \begin{matrix} = R(x) \\ = Q(x) \end{matrix}$$

How many constants do you need in the decomposition?

$$\frac{R(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x-1} + \frac{Fx+G}{x^2+x+1} + \frac{Hx+I}{x^2+1}$$

$$+ \frac{Jx+K}{(x^2+1)^2} + \frac{Lx+M}{(x^2+1)^3}$$

13 constants

7.5 Strategy for Integration

1) Simplify the integrand if possible

2) Look for an obvious substitution

3) Classify the integrand:

a) Trig integral $\int \cos^3(x)\sin^5(x)\,dx$

b) Rational fn $\int \frac{P(x)}{Q(x)}\,dx$

c) Radicals $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$

4) Keep in mind that u-subs and integ. by parts are general techniques that can be applied in many situations

5) If you still don't have to answer, repeat from step 1.

e.g. $\int \frac{-13e^{2x}-72e^x}{e^{2x}+13e^x+36}\,dx$ $\quad u=e^x$, $du=e^x\,dx$

$$= \int \frac{(-13e^x-72)e^x\,dx}{e^{2x}+13e^x+36} = \int \frac{-13u-72}{u^2+13u+36}\,du$$

The integrand is a rational function now so we can use techniques from the previous.