$\sqrt{a^2-x^2}$	$x = a\sin t$ or $x = a\cos t$
$\sqrt{a^2+x^2}$	$x = a\tan t$
$\sqrt{x^2-a^2}$	$x = a\sec t$

Q) What to do if we have an expression that's not of these forms?

Example $\displaystyle\int \frac{1}{\sqrt{1+(7x-8)^2}}\,dx$

make u-sub

$u = 7x-8$

$du = 7\,dx$

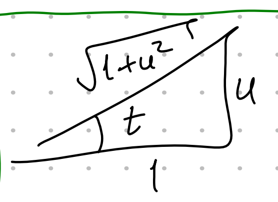
$$= \frac{1}{7}\int \frac{1}{\sqrt{1+u^2}}\,du$$

$\boxed{u = \tan t}$

$du = \sec^2 t\,dt$

$$= \frac{1}{7}\int \frac{\sec^2 t\,dt}{\sqrt{1+\tan^2 t}} = \frac{1}{7}\int \frac{\sec^2 t\,dt}{\sqrt{\sec^2 t}}$$

$$= \frac{1}{7}\int \frac{\sec^{\cancel{2}} t\,dt}{\cancel{\sec t}} = \frac{1}{7}\int \sec t\,dt = \frac{1}{7}\ln|\sec t + \underbrace{\tan t}_{u}| + C$$



$u = \tan t$ $\longrightarrow$ $\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{\sqrt{1+u^2}}}$

$\sec t = \sqrt{1+u^2}$

$$= \frac{1}{7}\ln|\sqrt{1+u^2} + u| + C$$

$$= \frac{1}{7}\ln|\sqrt{1+(7x-8)^2} + 7x-8| + C$$

Example $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

a^2+u^2
a^2-u^2
u^2-a^2

$3-2x-x^2$ We complete it to a square!

$$= 4-1-2x-x^2 = 4-(1+2x+x^2)$$
$$= 4-(x+1)^2$$

$$\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$u = x+1$ $du = dx$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$u = 2\sin t$ or $u = 2\cos t$

7.4 Integration of Rational Functions by Partial Fractions

$$\text{Rational Function} = \frac{\text{Polynomial}}{\text{Polynomial}}$$

$$\int \frac{1}{ax+b}\,dx$$

$u = ax+b$

$du = a\,dx$

$$= \frac{1}{a}\int \frac{1}{u}\,du = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C$$

e.g. $\displaystyle \frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)}$

$$= \frac{2x+4-x+1}{x^2+2x-x-2} = \frac{x+5}{x^2+x-2}$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

If $\deg P \geq \deg Q$ then we can do a polynomial long division to get to the RHS where $\deg R < \deg Q$

e.g. $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{r} x^2+x+2 \\ x-1 \overline{\big)\, x^3 \qquad +x} \\ -\ \underline{x^3-x^2} \\ x^2+x \\ -\ \underline{x^2-x} \\ 2x \\ -\ \underline{2x-2} \\ 2 \end{array}$$

$$\longrightarrow \quad \frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

$$\int \frac{x^3+x}{x-1}\,dx = \int\left(x^2+x+2+\frac{2}{x-1}\right)dx = \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1| + C$$

So we can just focus on how to integrate $\frac{R(x)}{Q(x)}$ where $\deg R < \deg Q$

First we factorize $Q(x)$ into linear or quadratic terms: $(ax+b)$ or (ax^2+bx+c)
$b^2-4ac<0$

Then $\frac{R(x)}{Q(x)}$ can be expressed as a sum of terms of the form $\frac{A}{(ax+b)^i}$, $\frac{Ax+B}{(ax^2+bx+c)^i}$

Case I: $Q(x)$ is a product of distinct linear factors

e.g.: $Q(x)=(x+1)(x+2)(2x+3)$ ✓
$Q(x)=(x+1)^2(x-5)$ ✗
$Q(x)=(x+1)(2x+2)$ ✗
$=2(x+1)^2$

Say
$Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$
Then
$$\frac{R(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}$$

Example $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$ ← R, ← Q

deg(R) < deg(Q) so polynomial division is not necessary

$Q(x)=x(2x^2+3x-2)$
$\begin{array}{ccc} 2x & & -1 \\ x & \times & +2 \end{array}$
$=x(2x-1)(x+2)$ (distinct roots $0, \frac{1}{2}, -2$) so distinct factors

Thus
$$\frac{R(x)}{Q(x)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}=\frac{x^2+2x-1}{2x^3+3x^2-2x}$$

$$\frac{A(2x-1)(x+2)+B(x)(x+2)+C(x)(2x-1)}{\cancel{x(2x-1)(x+2)}}=\frac{x^2+2x-1}{\cancel{2x^3+3x^2-2x}}$$

$A(2x-1)(x+2)+B(x)(x+2)+C(x)(2x-1)=x^2+2x-1$

$x=0$
$A(-1)(2)+0+0=-1$
$-2A=-1 \qquad A=\frac{1}{2}$

$x=-2$
$C(-2)(-5)=4-4-1$
$C(10)=-1 \qquad C=-1/10$

$x=\frac{1}{2}$
$B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)=\frac{1}{4}+1-1$
$\frac{5B}{4}=\frac{1}{4} \qquad B=\frac{1}{5}$

$$\int\frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx=\int\left(\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\right)dx$$
$$=A\ln|x|+B\frac{1}{2}\ln|2x-1|+C\ln|x+2|+K$$
$$=\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K$$

Example $\int \frac{1}{x^2-4}\,dx$ $\qquad Q(x) = x^2-4 = (x-2)(x+2)$

$$\frac{1}{x^2-4} = \frac{A}{\cancel{x-2}}^{(x-2)(x+2)} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-2)$$

$x=2 \rightarrow 1 = 4A \qquad A = \frac{1}{4}$

$x=-2 \rightarrow 1 = -4B \qquad B = -\frac{1}{4}$

$$\int \frac{1}{x^2-4}\,dx = \int \frac{A}{x-2}\,dx + \int \frac{B}{x+2}\,dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

Case II: $Q(x)$ has repeated linear factors.

e.g. $Q(x) = (x+5)^{3}(x+1)^{2}(4x-3)$ ✓

$Q(x) = (x+1)(2x+2) = 2(x+1)^{2}$ ✓

If $Q(x)$ has a repeated linear factor $(ax+b)^r$ then we will use terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_r}{(ax+b)^r}$$

(If $r=1$, we are in the prev. case.)

e.g.

$$\frac{x^3-x+1}{\underbrace{x^2}_{\text{2 terms}}\underbrace{(x-1)^3}_{\text{3 terms}}} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Example $\int \frac{4x}{x^3-x^2-x+1}\,dx$

$Q(x) = x^3-x^2-x+1 \qquad Q(1) = 0$

$\qquad\qquad\qquad\qquad\qquad \Downarrow$

$Q(x) = x^2(x-1) - (x-1) \qquad Q(x)$ is divisible by $(x-1)$

$= (x-1)(x^2-1) = (x-1)(x-1)(x+1)$

$= (x-1)^2(x+1)$ so

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = A\int \frac{1}{x-1}\,dx + B\int \underbrace{(x-1)^{-2}\,dx}_{\text{u-sub}} + C\int \frac{1}{x+1}\,dx$$

$$= A\ln|x-1| - B(x-1)^{-1} + C\ln|x+1| + K$$

NO CLASS ON WEDNESDAY (March 3)