Example $\int \sec^3 x\, dx$ $\qquad u = \sec x \qquad dv = \sec^2 x\, dx$

$du = \sec x \tan x\, dx \qquad v = \tan x$

$$= \int \sec x \sec^2 x\, dx$$

$$= \int u\, dv = uv - \int v\, du = \tan x \sec x - \int \tan^2 x \sec x\, dx$$

$$= \tan x \sec x - \int (\sec^2 x - 1)\sec x\, dx$$

$\boxed{\sec^2 x = 1 + \tan^2 x \quad \tan^2 x = \sec^2 x - 1}$

$$= \tan x \sec x - \int (\sec^3 x - \sec x)\, dx$$

$$= \tan x \sec x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$= \tan x \sec x - \int \sec^3 x\, dx + \ln|\sec x + \tan x| + C$$

$\boxed{\text{Set } I = \int \sec^3 x\, dx}$ So far,

$$I = \tan x \sec x - I + \ln|\sec x + \tan x| + C$$

$$2I = \tan x \sec x + \ln|\sec x + \tan x| + C$$

$$I = \frac{1}{2}(\tan x \sec x + \ln|\sec x + \tan x| + C)$$

Sometimes we use the following identities to integrate:

(a) $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

(b) $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

(c) $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

Example Evaluate

$$\int \sin(4x)\cos(5x)\, dx$$

$$= \frac{1}{2}\int (\sin(4x-5x) + \sin(4x+5x))\, dx$$

$$= \frac{1}{2}\int \sin(-x)\, dx + \frac{1}{2}\int \sin(9x)\, dx$$

$$= \frac{1}{2}\cos(-x) - \frac{1}{2}\frac{\cos(9x)}{9} + C$$

7.3 Trigonometric Substitution

u-sub: $\qquad u = g(x) \qquad du = g'(x)\, dx$

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

going backwards is inverse substitution

Expression	Subs	Identity
$\sqrt{a^2 - x^2}$ $\sqrt{1-\sin^2\theta}$	$x = a\sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $x = a\cos\theta \quad 0 \le \theta \le \pi$	$1 - \sin^2\theta = \cos^2\theta$ $1 - \cos^2\theta = \sin^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta \quad 0 \le \theta < \frac{\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx$ $\qquad x = 3\cos\theta$

$dx = -3\sin\theta\, d\theta$

$$= -3\int \frac{\sqrt{9 - (3\cos\theta)^2}}{(3\cos\theta)^2}\sin\theta\, d\theta$$

$\boxed{\sqrt{ab} = \sqrt{a}\sqrt{b}}$

$$= -3\int \frac{\sqrt{9 - 9\cos^2\theta}}{9\cos^2\theta}\sin\theta\, d\theta = -3\int \frac{\sqrt{9(1-\cos^2\theta)}}{9\cos^2\theta}\sin\theta\, d\theta$$

$$= -\cancel{3}\int \frac{\cancel{3}\sqrt{1-\cos^2\theta}}{\cancel{9}\cos^2\theta}\sin\theta\, d\theta = -\int \frac{\sqrt{\sin^2\theta}}{\cos^2\theta}\sin\theta\, d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$$

$\sqrt{a^2} = |a|$

$\sqrt{\sin^2\theta} = |\sin\theta|$

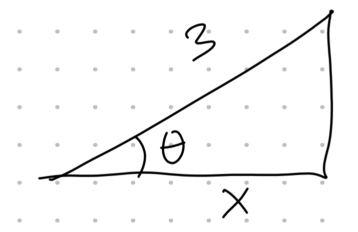
$= \sin\theta$ since $0 \le \theta \le \pi$

$$= -\int \tan^2\theta\, d\theta$$

$$= -\int (\sec^2\theta - 1)\, d\theta = -\tan\theta + \theta + C$$

We need to back substitute! $\qquad x = 3\cos\theta$

$\Rightarrow \cos\theta = \frac{x}{3} \longrightarrow \theta = \cos^{-1}\left(\frac{x}{3}\right)$

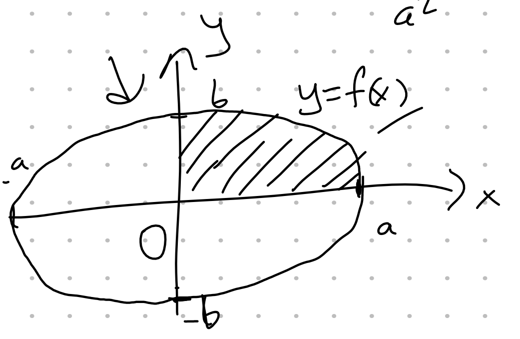


$\rightarrow \tan\theta = \frac{\sqrt{9-x^2}}{x}$

$$= -\frac{\sqrt{9-x^2}}{x} + \cos^{-1}\left(\frac{x}{3}\right) + C$$

Example Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

Area of Ellipse $= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = 4\int_0^a b\sqrt{\frac{1}{a^2}(a^2-x^2)}\,dx$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx \qquad x = a\sin\theta$$

$$dx = a\cos\theta\, d\theta$$

$$= \frac{4b}{\cancel{a}}\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\;\cancel{a}\cos\theta\, d\theta$$

$x = a \rightarrow a\sin\theta = a$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

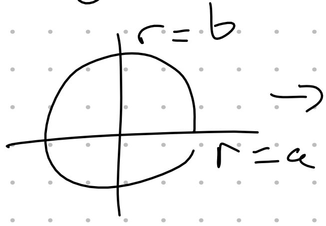
$x = 0 \rightarrow a\sin\theta = 0$, $\theta = 0$

$$= 4ab\int_0^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta = 4ab\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\, d\theta = 2ab\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 2ab\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + \underbrace{0}_{\sin(2(\pi/2))} - 0\right)$$

$$= \pi ab$$



$\rightarrow \pi r^2$

Example $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $\qquad (\tan\theta)^2 + 1 = \sec^2\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\, 2\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\, d\theta = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\, d\theta$$

$$= \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\, d\theta = \frac{1}{4}\int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\tan^2\theta\, \cancel{\sec\theta}}$$

$$= \frac{1}{4}\int \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, \frac{1}{\cancel{\cos\theta}}\, d\theta \qquad \boxed{\sec\theta = \frac{1}{\cos\theta} \quad \tan\theta = \frac{\sin\theta}{\cos\theta}}$$

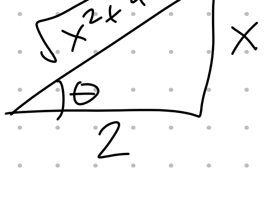
$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta \qquad u = \sin\theta$$

$$du = \cos\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{1}{u^2}\, du = \frac{1}{4}\int u^{-2}\, du = \frac{-1}{4}u^{-1} + C$$

$$= -\frac{1}{4}\,\frac{1}{\sin\theta} + C = \frac{-1}{4\sin\theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

$x = 2\tan\theta \qquad \tan\theta = \frac{x}{2}$



$\rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}}$