Example

Calculate $\int_0^1 \tan^{-1}(x)\,dx$

$u=\tan^{-1}(x)$ $dv=dx$

$du=\frac{1}{1+x^2}dx$ $v=x$

$$\int_0^1 u\,dv = uv\Big|_0^1 - \int_0^1 v\,du$$

$$= x\tan^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx$$

$u=1+x^2$

$du=2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \tan^{-1}(1) - 0 - \frac{1}{2}\int_1^2 \frac{1}{u}du$$

$x=0 \to u=1$

$x=1 \to u=2$

$$= \frac{\pi}{4} - \frac{1}{2}\ln|u|\Big|_1^2 = \frac{\pi}{4} - \frac{1}{2}\left(\ln(2) - \underbrace{\ln(1)}_{0}\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

7.2 Trigonometric Integrals

Example $\int \cos^3 x\,dx$

$\cancel{u=\cos x}$

$\cancel{du=-\sin x\,dx}$

$$\int \underbrace{\cos^2 x}_{1-\sin^2 x}\ \underbrace{\cos x\,dx}_{du}$$

$u=\sin x$

$du=\cos x\,dx$

$$\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1-\sin^2 x$$

$$\int (1-\sin^2 x)\cos x\,dx = \int (1-u^2)\,du$$

$$\boxed{\begin{aligned}\cos^4 x &= (\cos^2 x)^2\\ &= (1-\sin^2 x)^2\end{aligned}}$$

Example $\int \sin^5 x\cos^2 x\,dx$

$\cancel{u=\sin x}$

$\cancel{du=\cos x\,dx}$

$\cancel{\int \sin^5 x\cos x\ \underbrace{\cos x\,dx}}$

~~hard to express~~ in terms of $\sin x$

$$= \int \underbrace{\sin^4 x}_{(1-u^2)^2}\ \underbrace{\cos^2 x}_{u^2}\ \underbrace{\sin x\,dx}_{(-du)}$$

$u=\cos x$

$du=-\sin x\,dx$

$$\sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2 = (1-u^2)^2$$

$$= -\int (1-u^2)^2 u^2\,du = -\int (1-2u^2+u^4)u^2\,du$$

$$= -\int (u^2 - 2u^4 + u^6)\,du = -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= -\left(\frac{\cos^3 x}{3} - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x\right) + C$$

- When evaluating $\int \sin^m(x)\cos^n(x)\,dx$, if the power of sin ($m$) is odd, $u=\cos x$ and if the power of cos (n) is odd $u=\sin x$ substitution should give us the answer.
- What to do if we have $\int \sin^{\text{even}}(x)\cos^{\text{even}}(x)\,dx$?

We will use the following half-angle formulas

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x)$$

Example Evaluate $\int_0^{\pi} \sin^2 x\,dx$

$$= \frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\Big|_0^{\pi}$$

$$\boxed{\begin{aligned}&\int \cos x\,dx = \sin x + C\\ &\int \cos 5x\,dx = \frac{\sin 5x}{5} + C\\ &u=5x \to du = 5dx\end{aligned}}$$

$$= \frac{1}{2}\left[\pi - \frac{\sin(2\pi)}{2} - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$\boxed{\int_0^{\pi}\sin(2x)\,dx = 0}$$

$$= \frac{1}{2}\pi$$

Example $\int \sin^4 x\,dx$

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1}{2}(1-\cos 2x)\right)^2$$

$$= \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right)$$

$$\int \sin^4 x\,dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\left(x - \cancel{2}\frac{\sin 2x}{\cancel{2}} + \int \cos^2 2x\,dx\right) = \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{4}\int \cos^2 2x\,dx$$

$$\boxed{\cos^2 2x = \frac{1}{2}(1 + \cos 4x)} \quad = \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{8}\int (1 + \cos 4x)\,dx$$

$$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{8}\left(x + \frac{\sin 4x}{4}\right) + C$$

$(\tan x)' = \sec^2 x \qquad (\sec x)' = \tan x \sec x \qquad \sec^2 x = 1 + \tan^2 x$

Example

$$\int \tan^6 x \sec^4 x\,dx$$

$u = \sec x$

$du = \tan x \sec x\,dx$

$$= \int \tan^5 x \underbrace{\sec^3 x}_{u^3} \underbrace{\tan x \sec x\,dx}_{du}$$

hard to express in terms of $\sec x$

$$= \int \underbrace{\tan^6 x}_{u^6} \underbrace{\sec^2 x}_{1+\tan^2 x = 1+u^2} \underbrace{\sec^2 x\,dx}_{du}$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^6(1+u^2)\,du = \int (u^6 + u^8)\,du = \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example $\int \tan^5 t \sec^7 t\,dt$

$u = \sec t$

$du = \tan t \sec t\,dt$

$$\boxed{\begin{aligned}\sec^2 x &= 1 + \tan^2 x\\ \tan^2 x &= \sec^2 x - 1\end{aligned}}$$

$$= \int \underbrace{\tan^4 t}_{(\tan^2 t)^2 = (\sec^2 t - 1)^2 = (u^2-1)^2} \underbrace{\sec^6 t}_{u^6} \underbrace{\tan t \sec t\,dt}_{du}$$

$$= \int (u^2-1)^2 u^6\,du = \int (u^4 - 2u^2 + 1)\,u^6\,du$$

$$= \int (u^{10} - 2u^8 + u^6)\,du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\sec^{11} t}{11} - \frac{2}{9}\sec^9 t + \frac{1}{7}\sec^7 t + C$$

- When evaluating $\int \tan^{\text{any}} x \sec^{\text{even}} x\,dx$

  $u = \tan x \qquad du = \sec^2 x\,dx$ will work

- When evaluating $\int \tan^{\text{odd}} x \sec^{\text{any}} x\,dx$

  $u = \sec x \qquad du = \tan x \sec x\,dx$ will work

- What to do for $\int \tan^{\text{even}}(x) \sec^{\text{odd}}(x)\,dx$?

  No short answer. The following formulas are useful

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\boxed{\int \sec x\,dx = \int \sec x\,\frac{(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx}$$

$u = \sec x + \tan x$

$du = (\sec^2 x + \sec x \tan x)\,dx$

$$= \int \frac{du}{u} = \ln|u| + C$$

We might also use integration by parts or u-subs. along the way.

Example $\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx$

$$= \int \tan x(\sec^2 x - 1)\,dx = \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$

$$\int u\,du \qquad -\ln|\sec x|$$

$$\frac{u^2}{2} = \frac{\tan^2 x}{2} \qquad \frac{\tan^2 x}{2} - \ln|\sec x| + C$$