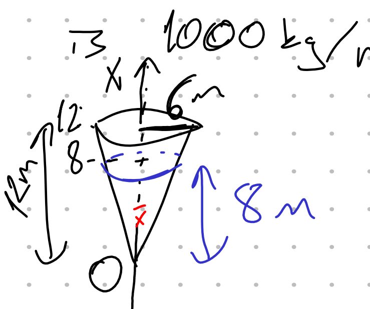
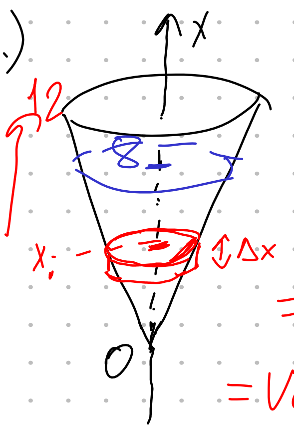
Example A tank has the shape of an inverted circular cone with height 12 m and base radius 6 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m³.)





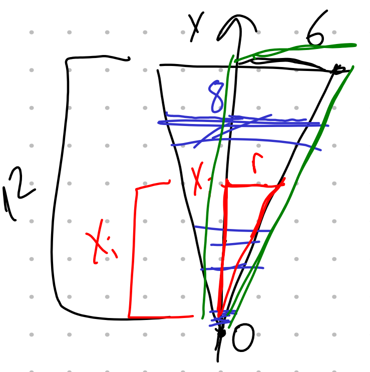
W_i = Force × displacement
$= \text{Weight} \times (12 - x_i)$
$= mg \times (12 - x_i)$
$= \text{Vol} \times \text{density} \times g \times (12 - x_i)$

$(g = 9.8)$

$\left(\text{density} = \dfrac{\text{mass}}{\text{Volume}}\right) \to \text{mass} = \text{density} \times \text{Volume}$

Volume = base Area × height height $= \Delta x$

Area $= \pi r^2$



$$\xrightarrow{} \quad \frac{x_i}{r} = \frac{12}{6} \Rightarrow \frac{r}{x_i} = \frac{6}{12} = \frac{1}{2}$$

h → x_i, b → r

$$r = \frac{x_i}{2}$$

$$W_i = \text{Vol} \times \text{density} \times g \times (12 - x_i)$$
$$= \pi r^2 h \cdot 1000 \cdot 9.8 \cdot (12 - x_i)$$
$$= \frac{\pi x_i^2}{4} \Delta x \;\; 9800\,(12 - x_i)$$

$$\text{Total Work} = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \int_0^8 \frac{9800}{4}\pi\, x^2 (12 - x)\,dx$$

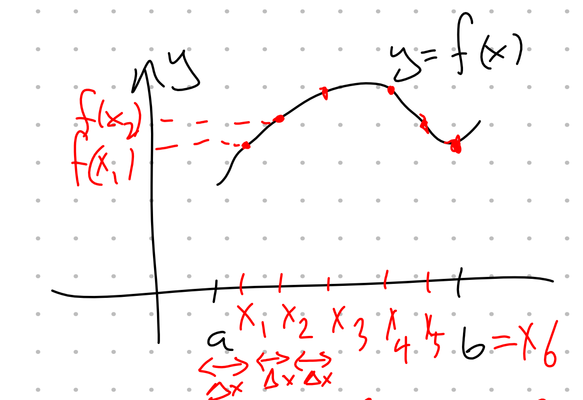
$$= \frac{9800\pi}{4} \int_0^8 (12x^2 - x^3)\,dx = \frac{9800\pi}{4}\left(\frac{\cancel{12}^{4} x^3}{\cancel{3}} - \frac{x^4}{4}\right)\Bigg|_0^8$$

$$= \frac{9800\pi}{4}\left(4(8^3) - \frac{8^4}{4}\right) \text{ J}$$

6.5 Average Value of a Function

$3, 5 \quad \dfrac{3+5}{2} \qquad\qquad \dfrac{1 + 10 + 9}{3} = \dfrac{20}{3}$

$a_1, a_2, \dots a_n \qquad \text{average} = \dfrac{a_1 + a_2 + \dots a_n}{n}$



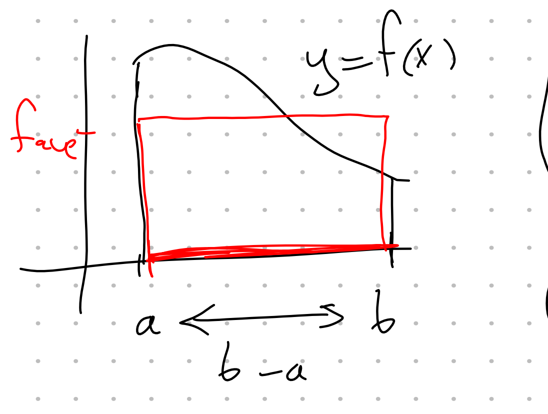
f_{ave} = the average value of f

$$\approx \frac{f(x_1) + f(x_2) + \dots + f(x_6)}{6}$$

$$f_{ave} = \lim_{n\to\infty} \frac{f(x_1) + \dots + f(x_n)}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \frac{\Delta x}{b-a}$$

$$\Delta x = \frac{b-a}{n} \rightarrow n = \frac{b-a}{\Delta x} \qquad = \frac{1}{b-a} \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x)\,\Delta x}$$



$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$$(b-a)\, f_{ave} = \int_a^b f(x)\,dx$$

base × height = Area under the curve

Example

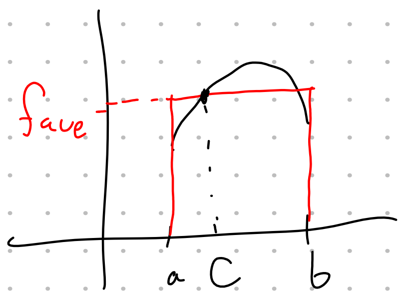
Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$

$$f_{ave} = \frac{1}{2-(-1)} \int_{-1}^{2} (1 + x^2)\,dx = \frac{1}{3}\left(x + \frac{x^3}{3}\right)\Bigg|_{-1}^{2}$$

$$= \frac{1}{3}\left(2 + \frac{8}{3}\right) - \frac{1}{3}\left(-1 - \frac{1}{3}\right) = 2$$

Theorem (The Mean Value Theorem for Integrals)
If f is continuous on $[a,b]$ then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



e.g. for $f(x) = 1+x^2$ on $[-1,2]$
$f_{ave} = 2$ by the prev. example

by the MVT for integrals, there is a c s.t.

$$f(c) = 2 \rightarrow \quad 1+c^2 = 2$$
$$c^2 = 2-1 = 1$$
$$c = \pm 1$$

Example Find the numbers b such that the average value of $f(x) = 2+6x-3x^2$ on the interval $[0,b]$ is equal to 4.

$$f_{ave} = 4 = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b}\int_0^b (2+6x-3x^2)\,dx$$

$$4 = \frac{1}{b}\left(2x + \frac{\cancel{6}^{3}x^2}{\cancel{2}} - \frac{\cancel{3}x^3}{\cancel{3}}\right)\Big|_0^b = \frac{1}{b}(2b+3b^2-b^3)$$

$$4 = 2+3b-b^2$$

$$b^2-3b+2 = 0$$
$$(b-1)(b-2) = 0 \qquad b = 1,2$$

In fact, you can verify that $\int_0^1 f(x)\,dx = 4 = 4(1-0)$

$$\int_0^2 f(x)\,dx = 8 = 4(2-0)$$

7.1 Integration by Parts

Product Rule $\quad (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$\int (f(x)g(x))'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

$$\underset{f(x)g(x) \text{ by the FTC}}{\parallel}$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by Parts

In Leibniz notation: $\quad u = f(x) \quad v = g(x)$
$\quad du = f'(x)dx \quad dv = g'(x)dx$

$$\int u\,dv = uv - \int v\,du$$

Example Find $\int x\sin x\,dx$ $\quad u = x \quad dv = \sin x\,dx$
$\quad du = dx \quad v = -\cos x$

$$\int u\,dv = uv - \int v\,du = -x\cos x - \int(-\cos x)\,dx$$
$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

Example $\int \ln x\,dx$ $\quad u = \ln x \quad dv = dx$
$\quad du = \frac{1}{x}dx \quad v = x$

$$= \int u\,dv$$
$$= uv - \int v\,du = x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx$$
$$= x\ln x - \int dx = x\ln x - x + C$$

Example $\int t^2 e^t\,dt$ $\quad u = t^2 \quad dv = e^t\,dt$
$\quad du = 2t\,dt \quad v = e^t$

$$= \int u\,dv = uv - \int v\,du = t^2e^t - \int e^t\,2t\,dt$$
$$= t^2e^t - 2\int te^t\,dt$$

(We need to apply

$= t^2e^t - 2\left(uv - \int v\,du\right)$ (Integration by parts twice)

$u = t \qquad dv = e^t dt$

$du = dt \qquad v = e^t$

$= t^2e^t - 2\left(te^t - \int e^t dt\right)$

$= t^2e^t - 2te^t + 2e^t + C$

Example $\int e^x \sin x\,dx$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x\,dx \qquad v = e^x$

$= uv - \int v\,du = e^x \sin x - \int e^x \cos x\,dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x\,dx \qquad v = e^x$

$= e^x \sin x - \left(uv - \int v\,du\right)$

$= e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x)\,dx\right)$

So far:

$\underbrace{\int e^x \sin x\,dx}_{I} = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\,dx}_{I}$

$\Rightarrow \quad 2I = e^x \sin x - e^x \cos x = e^x(\sin x - \cos x)$

$I = \frac{e^x}{2}(\sin x - \cos x)$

$\rightarrow \int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$

Examples next time.