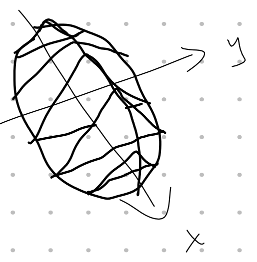
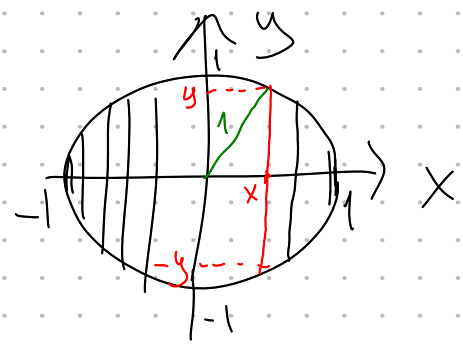
$V=\int_a^b A(x)\,dx$

Last time

Solids of revolution

Example The following figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.





$V=\int_a^b A(x)\,dx$

$a=-1 \quad b=1$

$A(x)=$ Area of an equilateral triangle of side $(y-(-y))$

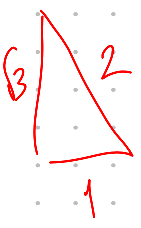
$=2y$

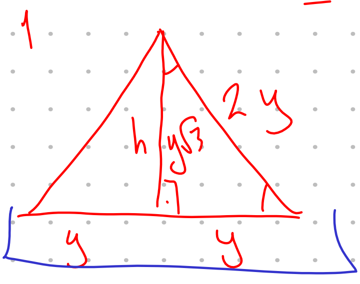
$=2\sqrt{1-x^2}$

$1^2=x^2+y^2 \Rightarrow y^2=1-x^2$

$y=\sqrt{1-x^2}$

$A=\frac{1}{\cancel{2}}\,b\,h \qquad \cancel{2}y \quad y\sqrt{3}$





$A=\frac{1}{\cancel{2}}(\cancel{2}y)\,y\sqrt{3}=y^2\sqrt{3}$

$=(1-x^2)\sqrt{3}$

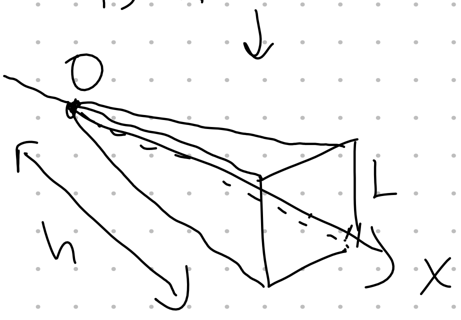
$(2y)^2=h^2+y^2 \rightarrow 4y^2=h^2+y^2$

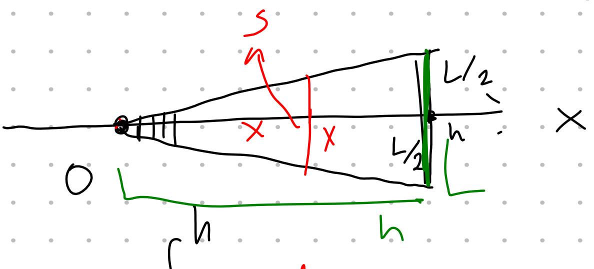
$3y^2=h^2 \qquad h=\sqrt{3}\,y$

$$V=\int_a^b A(x)\,dx=\int_{-1}^{1}\sqrt{3}\,(1-x^2)\,dx=2\sqrt{3}\int_0^1(1-x^2)\,dx$$

$$=2\sqrt{3}\left(x-\frac{x^3}{3}\right)\Big|_0^1=2\sqrt{3}\left(1-\frac{1}{3}\right)=\frac{4\sqrt{3}}{3}$$

Example Find the volume of a pyramid whose base is a square with side L and whose height is h.





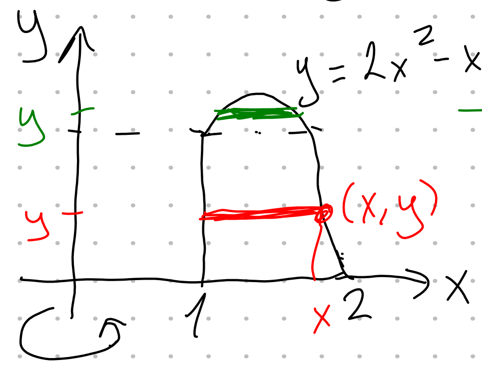
$V=\int_0^h A(x)\,dx$

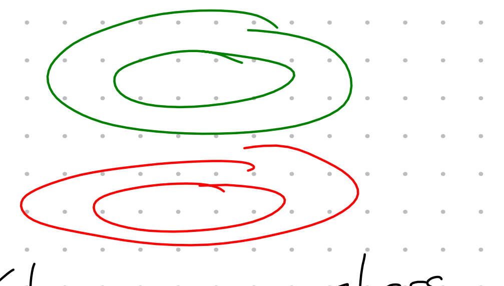
$\frac{x}{s}=\frac{h}{L} \rightarrow \frac{s}{x}=\frac{L}{h} \rightarrow s=\frac{xL}{h}$

$A(x)=s^2=\frac{x^2L^2}{h^2} \qquad V=\int_0^h \frac{x^2L^2}{h^2}\,dx=\frac{L^2}{h^2}\frac{x^3}{3}\Big|_0^h$

$=\frac{L^2h^{\cancel{3}}}{\cancel{h^2}\,3}=\frac{L^2h}{3}$

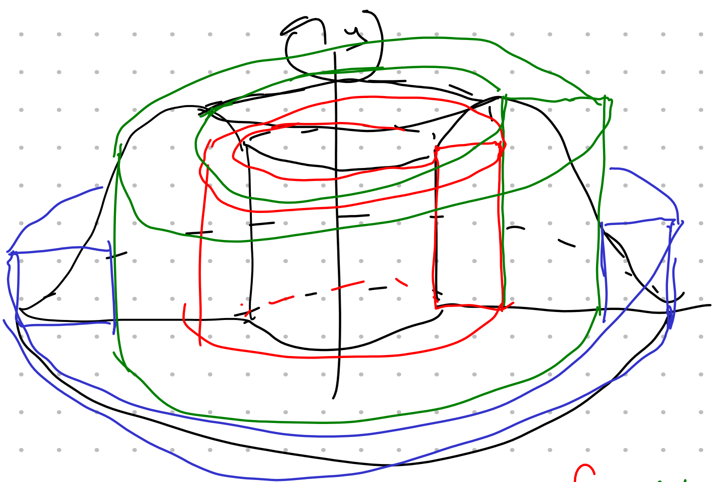
6.3 Volumes by Cylindrical Shells

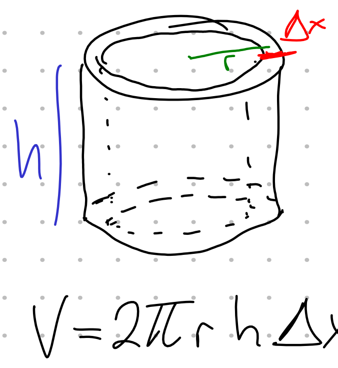




Slices are washers but it is difficult to find their radii.

Instead we will slice them "with cylinders".



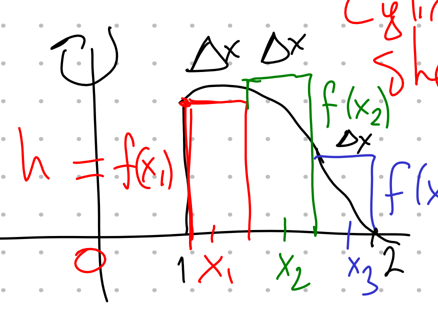


$V=2\pi r h \Delta x$

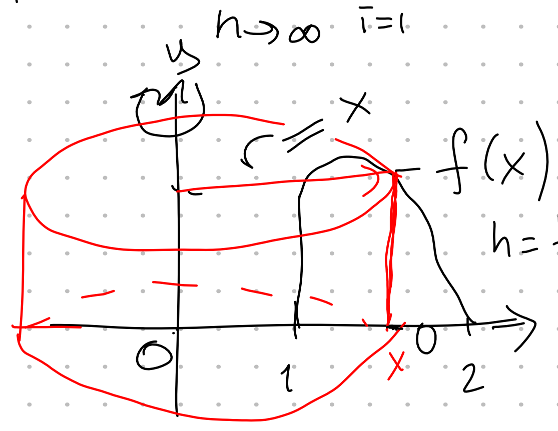
Volume $\approx$ Volume of Cylindrical Shell 1 + Volume of Cylindrical Shell 2 + Volume of Cylindrical Shell 3

$\| \qquad \|$

$2\pi x_1 f(x_1)\Delta x+2\pi x_2 f(x_2)\Delta x+2\pi x_3 f(x_3)\Delta x$



Volume $= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i f(x_i)\Delta x = \int_a^b 2\pi x f(x)\,dx$

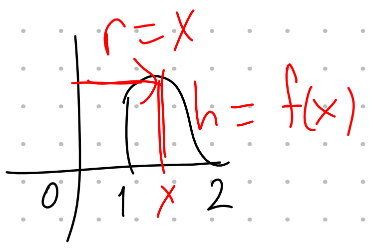


$V = \int_a^b A(x)\,dx$

$A(x)$ = Area of the cylinder passing through x

$A(x) = 2\pi r h = 2\pi x f(x)$ //

Example Find the volume of the solid obtained by rotating the region under $y = 2x^2 - x^3$ for $1 \le x \le 2$ about the y-axis.

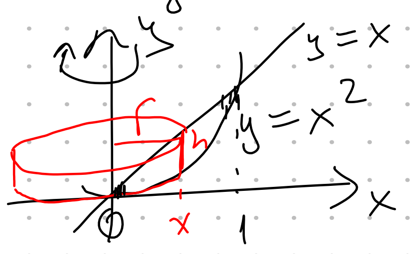


$V = \int_1^2 2\pi x f(x)\,dx$

$= 2\pi \int_1^2 (2x^3 - x^4)\,dx$

$= 2\pi\left(\frac{\cancel{2}x^4}{\cancel{4}_2} - \frac{x^5}{5}\right)\Big|_1^2 = 2\pi\left(8 - \frac{32}{5}\right) - 2\pi\left(\frac{1}{2} - \frac{1}{5}\right)$

Example Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



$r = x \quad h = x - x^2$

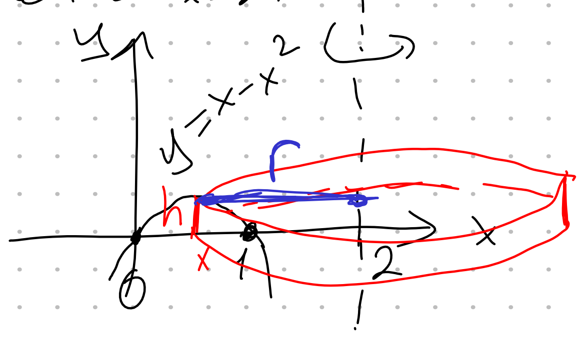
$A(x) = 2\pi r h$

$= 2\pi x(x - x^2)$

$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx$

$= 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 2\pi \frac{1}{12} = \frac{\pi}{6}$

Example Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.



$V = \int_a^b A(x)\,dx$

$r = 2 - x$

$h = x - x^2$

$a = 0 \quad b = 1$

$A(x) = 2\pi r h$

$= 2\pi(2 - x)(x - x^2)$

$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 (2x - 2x^2 - x^2 + x^3)\,dx$

$= 2\pi \int_0^1 (2x - 3x^2 + x^3)\,dx = 2\pi\left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{\cancel{3}x^3}{\cancel{3}} + \frac{x^4}{4}\right)\Big|_0^1$

$= 2\pi\left(1 - 1 + \frac{1}{4}\right) = \frac{\pi}{2}$