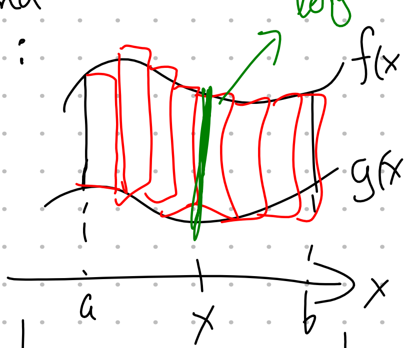
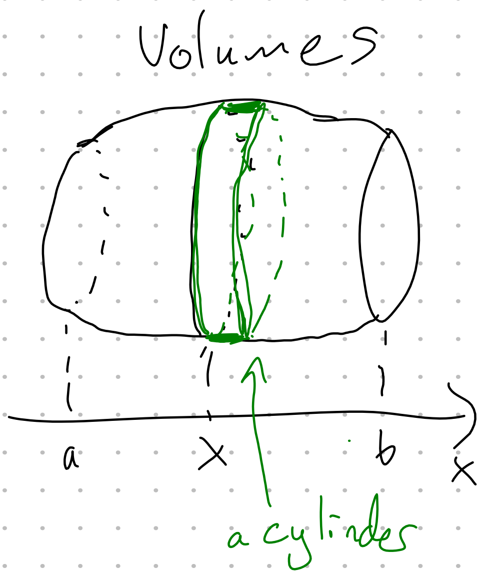
6.2 Volumes

To find Area:

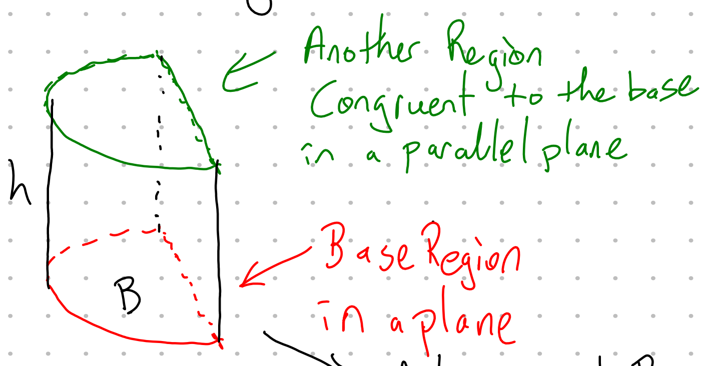


$$A = \int_a^b (f(x) - g(x))\,dx = \int_a^b \text{height}(x)\,dx$$

Volumes

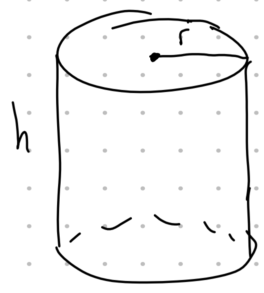


What are cylinders?



The set of all points in between these two regions is called a cylinder.

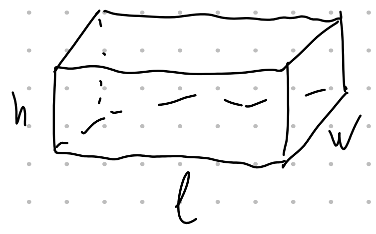
→ Volume = hB = height × Area of the base



"Circular" cylinder

(its base is a circle)

$V = h\pi r^2$ = height × Area of base



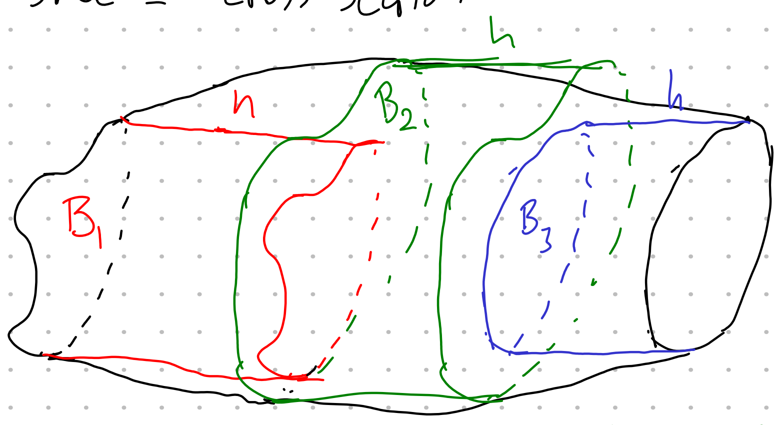
Rectangular box

box is a cylinder

$V = hlw = h(lw)$

= height × Area of base

slice = cross-section



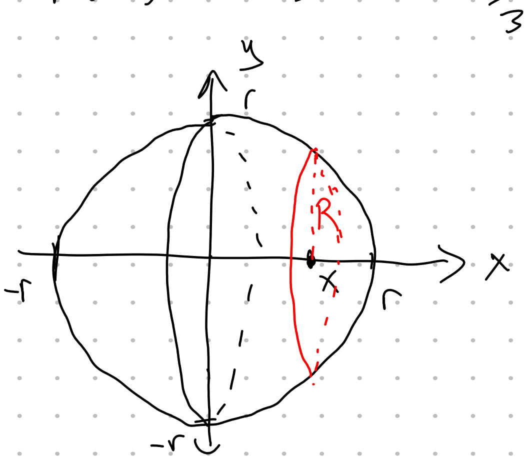
Volume ≃ Volume of Cylinder 1 + Volume of Cylinder 2 + Volume of Cylinder 3

$= hB_1 + hB_2 + hB_3$

__Definition__ let S be a solid that lies between $x=a$ and $x=b$. If the cross sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$ then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

__Example:__ Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$



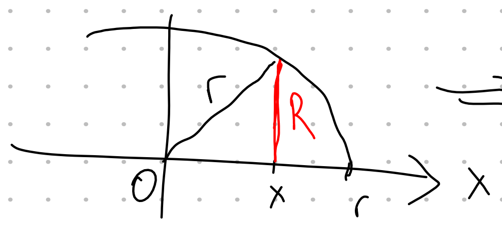
$$V = \int_a^b A(x)\,dx$$

$a = -r$ $\qquad b = r$

$A(x)$ = area of the slice at x

$= \pi R^2$

(R is a function of x)



$\Longrightarrow$ $r^2 = R^2 + x^2$

$R^2 = r^2 - x^2$

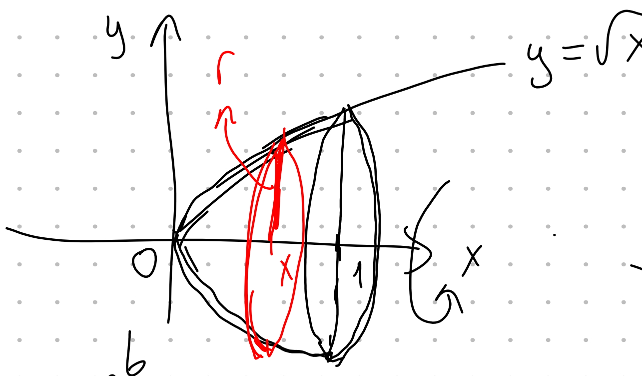
$R = \sqrt{r^2 - x^2}$

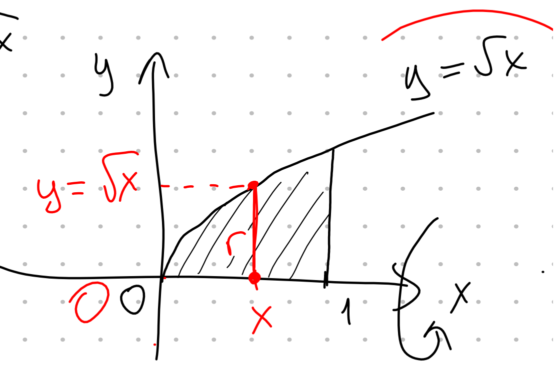
$$V = \int_a^b A(x)\,dx = \int_{-r}^{r} \pi R^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\int_{-r}^{r} r^2 - x^2\,dx$$

$$= \pi\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

$$= \pi\left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right) = \frac{4}{3}\pi r^3$$

__Example__ Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.





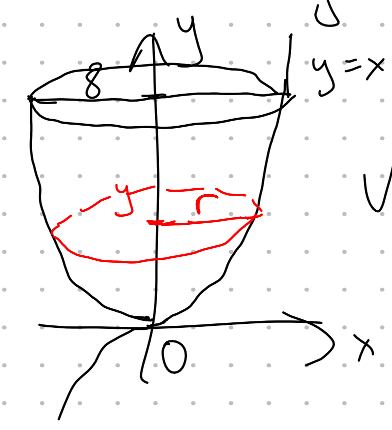
$$V = \int_a^b A(x)\,dx \qquad a=0 \quad b=1$$

$$A(x) = \pi r^2$$

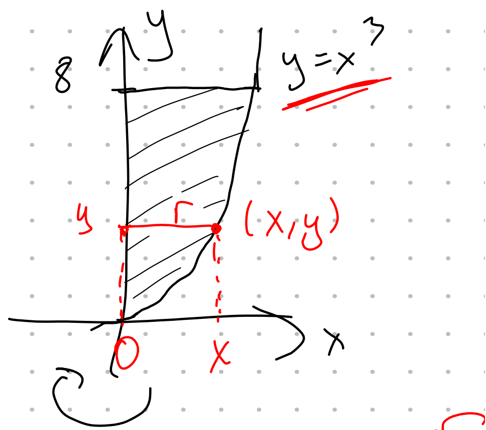
$r = \sqrt{x} - 0 = \sqrt{x}$
$(= y - 0)$

$$= \int_0^1 \pi(\sqrt{x})^2\,dx = \int_0^1 \pi x\,dx = \frac{\pi x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

Example Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.



$$V = \int_c^d A(y)\,dy$$



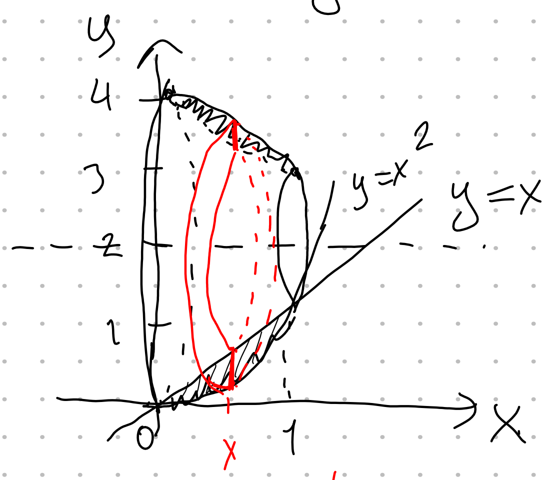
$r = x - 0 = x = \sqrt[3]{y}$

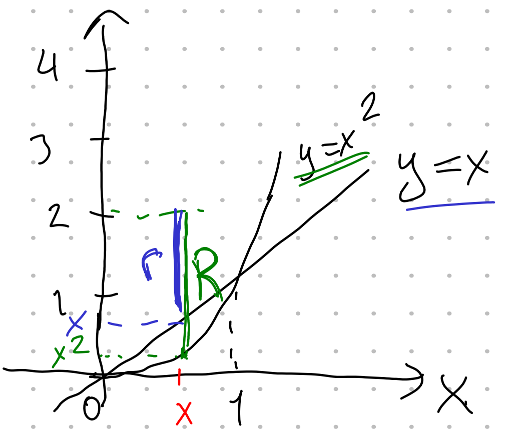
$c = 0 \quad d = 8 \quad A(y) = \pi r^2$

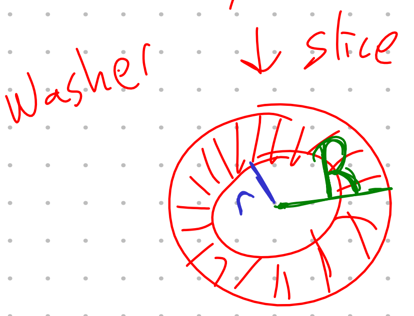
$$V = \int_0^8 \pi (y^{1/3})^2\,dy = \int_0^8 \pi y^{2/3}\,dy = \frac{3\pi}{5} y^{5/3}\Big|_0^8$$

$$= \frac{96\pi}{5}$$

Example The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume.







$$V = \int_a^b A(x)\,dx$$

$a = 0 \quad b = 1$

$r = 2 - x$
$R = 2 - x^2$

$$A(x) = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

$$V = \pi \int_0^1 (R^2 - r^2)\,dx = \pi \int_0^1 \left((2 - x^2)^2 - (2 - x)^2\right) dx$$

$$= \pi \int_0^1 \left(\cancel{4} - 4x^2 + x^4 - (\cancel{4} - 4x + x^2)\right) dx$$

$$= \pi \int_0^1 -5x^2 + 4x + x^4\,dx$$

$$= \pi\left(-\frac{5x^3}{3} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + \frac{x^5}{5}\right)\Big|_0^1 = \pi\left(-\frac{5}{3} + 2 + \frac{1}{5}\right)$$