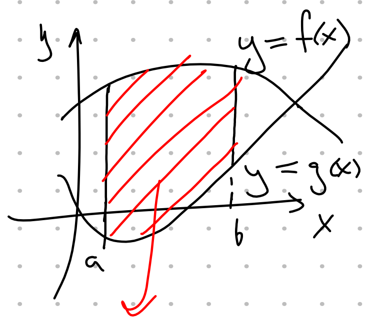
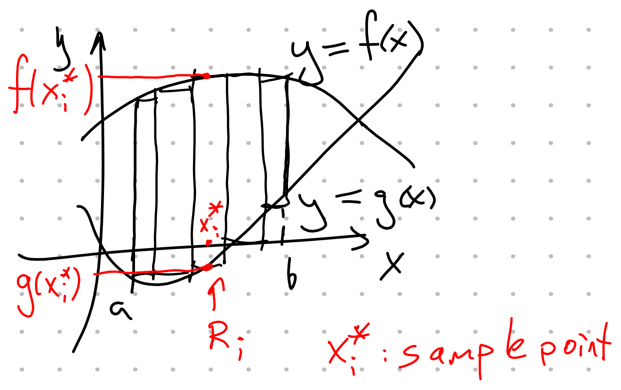
6.1 Areas Between Curves





Area below $y=f(x)$ above $y=g(x)$ for $a \le x \le b$ $\approx$ Area of these rectangles $= R_1 + R_2 + R_3 + \cdots R_n$

$= h_1 \Delta x + h_2 \Delta x + \cdots + h_n \Delta x$

$h_i = f(x_i^*) - g(x_i^*)$

$$= \sum_{i=1}^{n} h_i \Delta x = \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*) \right) \Delta x$$

$$\text{Exact Area} = \lim_{n \to \infty} \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*) \right) \Delta x$$

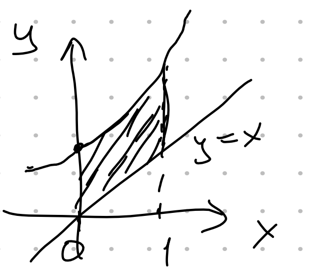
Recall $\displaystyle\int_a^b F(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i^*) \Delta x$

In our situation, $F(x) = f(x) - g(x)$

Thus, $\displaystyle \text{Exact Area} = \int_a^b (f(x) - g(x))\,dx$ — top function, bottom function

or $\displaystyle = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Example Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

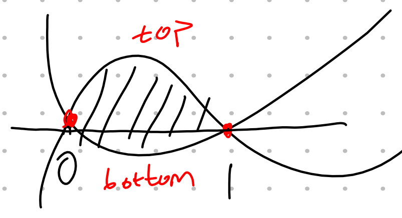


$$\text{Area} = \int_0^1 (e^x - x)\,dx = e^x - \frac{x^2}{2} \Big|_0^1$$

$$= e^1 - \frac{1}{2} - \left(e^0 - 0\right) = e - \frac{1}{2} - 1$$

$$= e - \frac{3}{2}$$

Example Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.



$x^2 = 2x - x^2$

$2x^2 - 2x = 0$

$x^2 - x = 0$

$x(x-1) = 0 \qquad x = 0, 1$

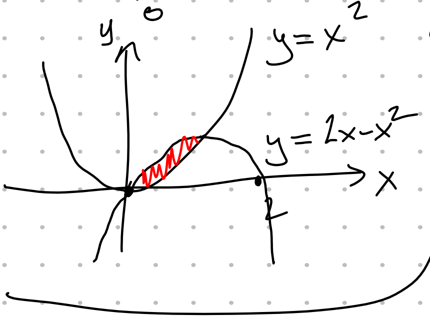
$f(x) = x^2 \qquad f\left(\frac{1}{2}\right) = \frac{1}{4}$

$g(x) = 2x - x^2 \qquad g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$

So g is the top curve.

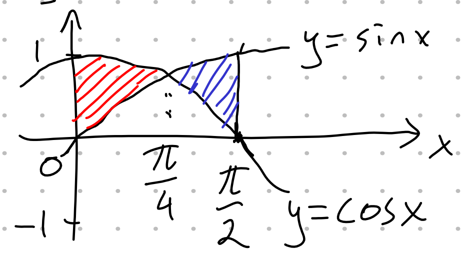
$$A = \int_0^1 (g(x) - f(x))\,dx = \int_0^1 (2x - x^2 - x^2)\,dx$$

$$= \int_0^1 (2x - 2x^2)\,dx = x^2 - \frac{2x^3}{3} \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$



The area between the curves $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ is $\displaystyle\int_a^b |f(x) - g(x)|\,dx$

Example Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$



$$\int_0^{\pi/2} |\sin x - \cos x|\,dx$$

we need to find where $\sin x - \cos x$ changes sign.

In other words $\sin x - \cos x = 0 \Rightarrow \sin x = \cos x$

$x = \frac{\pi}{4}$ $\qquad$ On $\left[0, \frac{\pi}{4}\right]$ $\cos x \ge \sin x$

$\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ $\sin x \ge \cos x$

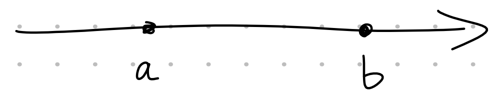
$$\int_0^{\pi/2} |\sin x - \cos x|\,dx = \int_0^{\pi/4} |\sin x - \cos x|\,dx + \int_{\pi/4}^{\pi/2} |\sin x - \cos x|\,dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= \sin x + \cos x \Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

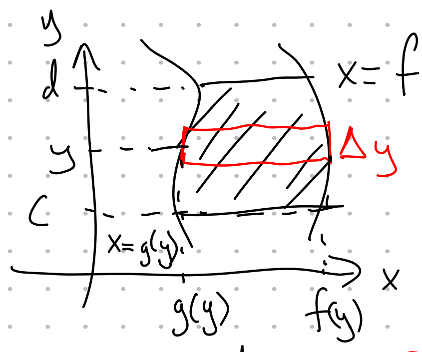
$$= \left(\frac{\sqrt2}{2} + \frac{\sqrt2}{2}\right) - (0+1) + (0-1) - \left(-\frac{\sqrt2}{2} - \frac{\sqrt2}{2}\right)$$

$$= 2\sqrt2 - 2$$



distance from a to b $= |b-a|$

$= b - a$ if $b \geq a$



height $= f(y) - g(y)$

(or $|f(y)-g(y)|$)

right − left

$$\text{Area} = \int_c^d (f(y) - g(y))\,dy$$

right left

Example Find the area enclosed by the line $y = x-1$ and the parabola $y^2 = 2x+6$

$x = y+1 = f(y)$ $\quad 2x = y^2 - 6$

$x = \frac12 y^2 - 3 = g(y)$

$y + 1 = \frac12 y^2 - 3 \rightarrow \frac12 y^2 - y - 4 = 0$

$y^2 - 2y - 8 = 0$

-4
$+2$

$(y-4)(y+2) = 0$

$y = -2, 4$

$f(0) = 1 \qquad g(0) = -3 \qquad f(y) \geq g(y)$ on $[-2, 4]$

right left

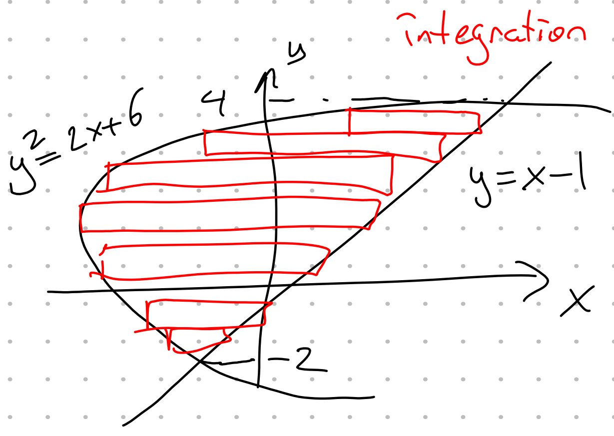
$$\text{Area} = \int_{-2}^{4} (f(y) - g(y))\,dy = \int_{-2}^{4} \left((y+1) - \left(\frac12 y^2 - 3\right)\right) dy$$

$$= \int_{-2}^{4} \left(-\frac12 y^2 + y + 4\right) dy = -\frac16 y^3 + \frac{y^2}{2} + 4y \Big|_{-2}^{4}$$

$$= -\frac16(64) + 8 + 16 - \left(\frac43 + 2 - 8\right) = 18$$

integration w.r.t. to y

we slice horizontally (perpendicular to the y-axis.)



integration wrt. x
slices are vertical

As we cross the blue line the bottom curve changes. That's why we need more than 1 integration