Example For what values of x does $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ CONV?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-3)^{n+1}}^{(x-3)}}{n+1}\cdot\frac{n}{\cancel{(x-3)^n}}\right| = \lim_{n\to\infty}|x-3|\underbrace{\left(\frac{n}{n+1}\right)}_{1}$$

$= |x-3|$

If $|x-3|<1$ then CONV $\left(\begin{array}{c}-1<x-3<1\\ 2<x<4\end{array}\right)$

If $|x-3|>1$ then DIV

by the ratio test.

If $|x-3|=1$, then $x-3=\pm1 \to x=3\pm1$, $x=2,4$

For $x=2$,

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(2-3)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \text{ CONV } \left(\text{"alternating harmonic series"}\right)$$

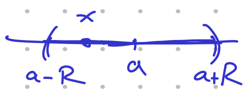
For $x=4$,

$$\sum_{n=1}^{\infty}\frac{(4-3)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n} \text{ DIV } \left(\text{"harmonic series"}\right)$$

So for x in $[2,4)$, the series is CONV.

Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

1) The series CONV only for $x=a$
2) // CONV for all x.
3) There is a positive number R such that the series is CONV if $|x-a|<R$ and DIV if $|x-a|>R$



R is called the radius of convergence.

The interval on which the series is CONV is called the interval of convergence.

We say $R=0$ in case (1)
$R=\infty$ in case (2)

Example Find the radius of convergence and interval of convergence of $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$ $\quad (x-(-2))^n$, $a=-2$, $x-a=x+2$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{(x+2)^{n+1}}}{\cancel{3^{n+2}}_{3}}\cdot\frac{\cancel{3^{n+1}}}{n\cancel{(x+2)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{1}{3}|x+2|\underbrace{\left(\frac{n+1}{n}\right)}_{1} = \frac{|x+2|}{3}$$

If $\frac{|x+2|}{3}<1$ the series is CONV.

$\boxed{|x+2|<3} = R$

$|x-a|<R$

If $|x+2|=3$, $x+2=\pm3$ $\quad x=-2\pm3$, $x=-5,1$

For $x=-5$, $\quad (-3)^n=(-1)^n 3^n$

$$\sum_{n=0}^{\infty}\frac{n(-5+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n(-1)^n\cancel{3^n}}{\cancel{3^{n+1}}} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{3}$$

$\frac{(-1)^n n}{3}$ is DIV

For $x=1$,

$$\sum_{n=0}^{\infty}\frac{n(1+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty}\frac{n\cancel{3^n}}{\cancel{3^{n+1}}} = \sum_{n=0}^{\infty}\frac{n}{3}$$

$\frac{n}{3}\to\infty$ (If $\lim a_n \neq 0$ then $\sum a_n$ DIV)

They are both DIV by the test for divergence.

The interval of conv: $(-5,1)$.

11.9 Representation of Functions as Power Series

$$\frac{1}{1-x} \overset{|x|<1}{=} 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty}x^n \quad (a=0)$$

a, $r=x$

(geometric series $r=x$ $|r|<1$ CONV, sum $= \frac{a}{1-r}$)

$|x|<1$

$|x-a|<R$

$R=1$ and $I=(-1,1)$

$$\boxed{\frac{1}{1-A} \overset{|A|<1}{=} 1+A+A^2+A^3+A^4+\cdots = \sum_{n=0}^{\infty}A^n \quad (\circledast)}$$

Example Express $\frac{1}{1+x^2}$ as the sum of a power series.

In $(\circledast)$, plug in $A=-x^2$

$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+x^8-\cdots = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$|A|<1 \to |-x^2|<1$

$|x^2|<1$ if and only if $|x|<1$

Thus, $R=1$, $I=(-1,1)$

Example Find a power series repr. of $\frac{1}{x+2}$

$$= \frac{1}{2+x} = \frac{1}{2}\cdot\frac{1}{\left(1+\frac{x}{2}\right)} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{x}{2}\right)} = \frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}-\cdots\right)$$

$1+A+A^2+A^3+\cdots$

(Plug in $A=\frac{-x}{2}$)

$|A|<1$, $\left|\frac{x}{2}\right|<1$, $|x|<2$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

$R=2$

$I=(-2,2)$

Example Find a power ser. repr. of $\frac{x^3}{x+2}$

$$x^3\cdot\frac{1}{x+2} = x^3\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{2^{n+1}}$$

$R=2$ $\quad I=(-2,2)$

Theorem If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a)+c_2(x-a)^2+\cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on $(a-R, a+R)$ and

1) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n\,c_n(x-a)^{n-1}$

2) $\int f(x)\,dx = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty}\frac{c_n(x-a)^{n+1}}{n+1}$

The radii of convergence is still R for both cases.

Example Express $\frac{1}{(1-x)^2}$ as a power series using (*)

$\frac{1}{1-A} \overset{(*)}{=} 1+A+A^2+A^3+\cdots$

$$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = (-1)(1-x)^{-2}\cdot(-1) = \frac{1}{(1-x)^2}$$

Using $A=x$ in (*) and differentiating both sides,

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = 0+1+2x+3x^2+\cdots = \sum_{n=0}^{\infty} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

$$= \sum_{n=0}^{\infty}(n+1)x^n$$

$R=1$ by the theorem.

Example Find a power ser. rep for $\ln(1+x)$

$$(\ln(1+x))' = \frac{1}{1+x} \overset{*\text{ with } A=-x}{=} 1-x+x^2-x^3+x^4-\cdots$$

$(R=1)$

Thus,

$$\ln(1+x) = \int(\ln(1+x))'dx = \int\frac{1}{1+x}dx = \int 1-x+x^2-x^3+x^4 \cdots\, dx$$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Plug in $x=0$, $\ln(1) = C+0-0+\cdots$

$0 = C$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (R=1)$$

$x \to x-1$

$$\ln(1+(x-1)) = \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

centered at $a=1$.

Example Do the same for $\tan^{-1}x$.

$$(\tan^{-1}x)' = \frac{1}{1+x^2} \overset{A=-x^2}{=} 1-x^2+x^4-x^6+\cdots \qquad (R=1)$$

Integrating both sides,

$$\tan^{-1}x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (R=1)$$

$x=0$,

$0 = \tan^{-1}0 = C+0+0\cdots \qquad C=0$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R=1)$$

Example Evaluate $\int\frac{1}{1+x^7}dx$ as a power series.

Approximate $\int_0^{0.5}\frac{1}{1+x^7}dx$ correct to within 10^{-7}.

$$\frac{1}{1+x^7} \overset{A=-x^7}{=} 1-x^7+x^{14}-x^{21}+x^{28}-\cdots = \sum_{n=0}^{\infty}(-1)^n x^{7n}$$

Integrate

$$\int\frac{1}{1+x^7}dx = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{7n+1}}{7n+1}$$

$$\int_0^{0.5}\frac{1}{1+x^7}dx = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \Bigg|_0^{0.5}$$

$$=\left(\cancel{C} + 0.5 - \frac{0.5^8}{8} + \frac{0.5^{15}}{15} \cdots \right) - \left(\cancel{C} + 0 - 0 + 0 - \cdots\right)$$

$$= 0.5 - \frac{0.5^8}{8} + \frac{0.5^{15}}{15} \cdots = \frac{1}{2} - \left(\frac{1}{2}\right)^8 \frac{1}{8} + \left(\frac{1}{2}\right)^{15} \frac{1}{15} \cdots$$

$a_n = \frac{(-1)^n}{2^{7n+1}(7n+1)}$ Alternating Series $|Error| = |R_n| \le b_{n+1}$

$b_{n+1} < 10^{-7}$

$\frac{1}{2^{29}(29)} < 10^{-7}$ so $n=4$ works and we should use the first 4 terms ($n=0,1,2,3$).