The final exam sign-up will open this Friday.
The final exam will be 90 minutes long.

More examples from Chp 11

Example $\sum_{n=1}^{\infty} n e^{-n^2} \longrightarrow \frac{x}{e^{x^2}}$ cont., positive, decreasing $\longrightarrow \int^{\infty} \frac{x}{e^{x^2}}dx \quad (u=x^2)$

(easy to integrate so we could use the Integral test)

$\sum \frac{n}{e^{n^2}}$

$f(x)=\frac{x}{e^{x^2}}$ show that $f'(x)<0$ for $x>a$

$\Rightarrow$ the Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n}$$

$$= \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{\cancel{e^{n^2}}}{e^{n^2+2n+1}} = \lim_{n\to\infty}\underbrace{\left(1+\frac{1}{n}\right)}_{\to 1}\cdot\underbrace{\frac{1}{e^{2n+1}}}_{\to 0} = 0<1$$

So $\sum n e^{-n^2}$ is CONV by the ratio test.

Example $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{n^4+1} \longrightarrow \left|(-1)^n\frac{n^3}{n^4+1}\right| = \frac{n^3}{n^4+1} \overset{LCT}{\sim} \frac{n^3}{n^4}=\frac{1}{n}$

but $\sum\frac{1}{n}$ is DIV. (p-series $p\le 1$)

(Then, $\sum\frac{n^3}{n^4+1}$ is DIV

(ABS.CONV $\Rightarrow$ CONV)

~~If the series of absolute values is DIV~~ we cannot conclude anything about the original series.

Alternating Series Test?

$(-1)^n$ positive

$b_n=|a_n|=\frac{n^3}{n^4+1} \longrightarrow f(x)=\frac{x^3}{x^4+1}$

- $b_{n+1}\le b_n$ ✓
- $\lim_{n\to\infty} b_n = 0$ ✓

$$f'(x)=\frac{3x^2(x^4+1)-x^3(4x^3)}{(x^4+1)^2}=\frac{3x^6+3x^2-4x^6}{(x^4+1)^2}$$

$$=\frac{-x^6+3x^2}{(x^4+1)^2}=\frac{\overset{+}{x^2}(-x^4+3)}{\underset{+}{(x^4+1)^2}}<0 \quad \text{for } x>10$$

$f$ is decreasing $\Rightarrow b_n$ is decreasing

Thus, by the AST, $\sum(-1)^n\frac{n^3}{n^4+1}$ is CONV.

Example $\sum_{k=1}^{\infty}\frac{2^k}{k!}$

(the Ratio Test)

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\cancel{2^{k+1}}^{\,2}}{\cancel{(k+1)!}_{(k+1)}}\cdot\frac{\cancel{k!}}{\cancel{2^k}} = \lim_{k\to\infty}\frac{2}{k+1}=0<1$$

Thus, $\sum\frac{2^k}{k!}$ is ABS.CONV and therefore CONV.

Example $\sum_{n=1}^{\infty}\frac{1}{2+3^n} \overset{L.C.T.}{\sim} \sum\frac{1}{3^n}=\sum\left(\frac{1}{3}\right)^n$ Geometric Series with $|r|=\frac{1}{3}<1$

CONV

Thus, CONV.

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{2+3^n}}{\frac{1}{3^n}} = \lim_{n\to\infty}\frac{\frac{1}{\cancel{3^n}(2/3^n+1)}}{\frac{1}{\cancel{3^n}}} = \lim_{n\to\infty}\frac{1}{\underset{0}{\frac{2}{3^n}}+1}=1 \ne 0$$

Thus, by the limit comp. test,

$\sum\frac{1}{2+3^n}$ is CONV since $\sum\frac{1}{3^n}$ is CONV.

Example $\sum_{n=1}^{\infty}\frac{n^{2n}}{(1+n)^{3n}} = \sum_{n=1}^{\infty}\frac{(n^2)^n}{((1+n)^3)^n}$ $\qquad \left((A^p)^r=A^{pr}\right)$

$$=\sum\left(\frac{n^2}{(1+n)^3}\right)^n$$

The root test:

$$\lim_{n\to\infty}|a_n|^{1/n}=\lim_{n\to\infty}\left(\left(\frac{n^2}{(1+n)^3}\right)^{n}\right)^{1/n}=\lim_{n\to\infty}\frac{n^2}{(1+n)^3}$$

$$=\lim_{n\to\infty}\frac{n^2}{\left(n\left(\frac{1}{n}+1\right)\right)^3}=\lim_{n\to\infty}\frac{\cancel{n^2}\ 1}{\cancel{n^3}_{\,n\to\infty}\underbrace{\left(\frac{1}{n}+1\right)^3}_{1}}=0<1$$

By the root test, $\sum\frac{n^{2n}}{(1+n)^{3n}}$ is ABS.CONV and therefore CONV.

Example $\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n} \le \cancel{\sum_{n=1}^{\infty}\frac{1}{1+2^n}}$ $\qquad -1\le\sin 2n\le 1$

$0\le|\sin 2n|\le 1$

$$\sum_{n=1}^{\infty}\frac{|\sin 2n|}{1+2^n}\le\sum\frac{1}{1+2^n}\sim\sum\frac{1}{2^n}=\sum\left(\frac{1}{2}\right)^n$$

Direct C.T. CONV LCT Geometric s. with $|r|=\frac{1}{2}<1$

CONV CONV

Thus $\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n}$ is ABS.CONV therefore CONV.

Example $\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{1}{n^2}\right)$

$\frac{1}{n^2}\to 0 \quad \to \cos 0 = 1$

AST: $b_n=|a_n|$

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\cos\left(\frac{1}{n^2}\right)=1\ne 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n b_n$ is DIV. $\sum a_n$ is DIV by the test for divergence

11.8 Power Series

$$\sum_{n=0}^{\infty}c_n x^n = c_0+c_1x+c_2x^2+c_3x^3+\cdots$$

$c_n$: coeffients

$$f(x)=\sum_{n=0}^{\infty}c_n x^n = c_0+c_1x+c_2x^2+\cdots$$

Domain of $f$ is the set of all $x$ for which the series converges.

e.g. if all $c_n = 1$

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \left(\begin{array}{l}\text{geometric series}\\ \text{with } r = x\end{array}\right)$$

it is CONV if and only if $|r| < 1$

Therefore the domain of $f$ is $(-1, 1)$.

More generally,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a powerseries centered at $a$.

Example For what values of $x$ is the series $\sum_{n=0}^{\infty} n!\, x^n$ CONV?

The Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(n+1)!}^{\,n+1}\, \cancel{x^{n+1}}^{\,x}}{\cancel{n!}\, \cancel{x^n}}\right| = \lim_{n\to\infty} \left((n+1)|x|\right)$$

If $|x| = 0$ (or $x = 0$) then $\lim_{n\to\infty} (n+1)\cdot 0 = \lim_{n\to\infty} 0 = 0$

If $|x| \neq 0$ then $\lim_{n\to\infty} (n+1)\underbrace{|x|}_{>0} = \infty$

Thus, the series is CONV for $x = 0$ but it is DIV for $x \neq 0$

Example $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)|x-3|$$

$$= |x-3| \overset{?}{<} 1$$

Next time!