Recall: $\sum a_n$, $\quad \sum |a_n| = |a_1| + |a_2| + \cdots$

Def^n If $\sum |a_n|$ is CONV, then $\sum a_n$ is called ABS. CONV.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is CONV (p-series with $p > 1$)

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is ABS. CONV

($\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is CONV by the alternating series test)

$\left(\begin{array}{l} b_n = |a_n| = \frac{1}{n^2} \\ b_{n+1} \le b_n \ ✓ \\ \lim_{n\to\infty} b_n = 0 \ ✓ \end{array} \right)$

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Recall that this series is also CONV by the alternating series

$\left(\begin{array}{l} b_n = \frac{1}{n} \\ b_{n+1} \le b_n \ ✓ \\ \lim_{n\to\infty} b_n = 0 \ ✓ \end{array} \right)$

However,

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is DIV (p-series $p \le 1$)

Thus, $\sum \frac{(-1)^{n-1}}{n}$ is not ABS. CONV.

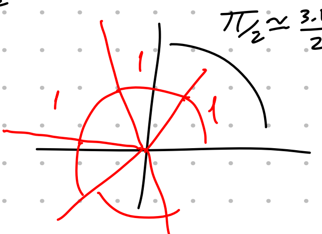
Def^n A series $\sum a_n$ is called conditionally convergent if it is CONV but not ABS. CONV.

In particular, $\sum \frac{(-1)^{n-1}}{n}$ is COND. CONV.

Thm If a series $\sum a_n$ is ABS CONV then it is CONV.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$

$\pi/2 \simeq \frac{3.14}{2}$



$\displaystyle\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$

By the direct comparison test,

$|\cos n| \le 1 \rightarrow \frac{|\cos n|}{n^2} \le \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ is CONV (p-series $p > 1$)

$\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is also CONV

$\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is ABS. CONV and therefore, CONV.

The Ratio Test

Consider $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

i) If $L < 1$, then $\sum a_n$ is ABS. CONV (and therefore CONV)

ii) If $L > 1$ or $L = \infty$, then $\sum a_n$ is DIV

iii) If $L = 1$, then the ratio test is inconclusive.

Note: $\sum \frac{1}{n^2}$ CONV

$\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2$

$\displaystyle = \lim_{n\to\infty} \left(\frac{1}{1 + \underbrace{\tfrac{1}{n}}_{0}} \right)^2 = 1$

$\left(\frac{a_{n+1}}{a_n} = a_{n+1} \cdot a_n^{-1} \right)$

$\sum \frac{1}{n}$ DIV $\qquad a_{n+1} \cdot a_n^{-1}$

$\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot n = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1$

Example $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

$|a_{n+1} \cdot a_n^{-1}|$

$\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{\underbrace{3^{n+1}}_{3^n \cdot 3}} \cdot \frac{\cancel{3^n}}{n^3} = \lim_{n\to\infty} \frac{1}{3} \left(\frac{n+1}{n} \right)^3$

$\displaystyle = \lim_{n\to\infty} \frac{1}{3} \left(1 + \underbrace{\tfrac{1}{n}}_{0} \right)^3 = \frac{1}{3} < 1$ Therefore $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is ABS. CONV (also CONV)

Example $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ $\qquad a_{n+1} \cdot a_n^{-1}$

$n! = 1 \cdot 2 \cdot 3 \cdots n$
$(n+1)! = 1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)$
$= n! \cdot (n+1)$

$\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{\cancel{n!}}{n^n}$

$\displaystyle = \lim_{n\to\infty} \frac{(n+1)^{\cancel{n+1}}}{\cancel{n+1}} \cdot \frac{1}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n \overset{(1^\infty)}{=} e > 1$

Thus, $\sum a_n$ is DIV by the ratio test.

$\displaystyle\lim_{n\to\infty} \boxed{\left(1 + \frac{1}{n}\right)^n} \overset{(1^\infty)}{=} \lim_{n\to\infty} \boxed{e^{\ln\left(1 + \frac{1}{n}\right)^n}}$ $\qquad \boxed{A = e^{\ln A}}$

$\displaystyle = \lim_{n\to\infty} \boxed{e^{n \ln\left(1 + \frac{1}{n}\right)}} = e^{\boxed{\lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right)}}$ $(\infty \cdot 0)$ $= e^1 = e$

$\displaystyle\lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \overset{\left(\frac{0}{0}\right)}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{1 + \frac{1}{x}} \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} = 1$

2nd Way

$\displaystyle a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} \ge n \to \infty$ as $n \to \infty$

Since $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ is DIV by the test for divergence.

The Root Test

$$\lim_{n\to\infty} |a_n|^{1/n} = L$$

i) If $L < 1$, then $\sum a_n$ is ABS. CONV (therefore CONV)

ii) If $L > 1$ or $L = \infty$ then $\sum a_n$ is DIV

iii) If $L = 1$, then the root test is inconclusive.

Example $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\left(\frac{2n+3}{3n+2}\right)^{\cancel{n}}\right)^{\cancel{1/n}} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}} = \frac{2}{3} < 1$$

Thus, $\sum a_n$ is ABS. CONV (and also CONV)

11.7 Examples from Chp 11 (So far)

Example $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ $\quad \displaystyle\lim_{n\to\infty} \frac{n-1}{2n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$

By the test for divergence, $\sum a_n$ is DIV.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

$$b_n = \frac{n^{3/2}}{n^3} = \frac{1}{n^{3-3/2}} = \frac{1}{n^{3/2}}$$

$\sum b_n = \sum \frac{1}{n^{3/2}}$ CONV (p-series $p>1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \underbrace{n^{3/2}}_{b_n^{-1}} \cdot \underbrace{\left(\frac{(n^3+1)^{1/2}}{3n^3+4n^2+2}\right)}_{a_n}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}\left(n^3\left(1+\frac{1}{n^3}\right)\right)^{1/2}}{n^3\left(3+\frac{4}{n}+\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(1+\frac{1}{n^3}\right)^{1/2}}{\cancel{n^3}\left(3+\frac{4}{n}+\frac{2}{n^3}\right)} = \frac{1}{3} \neq 0$$

By the limit comparison $\sum a_n$ is CONV since $\sum b_n$ is CONV.