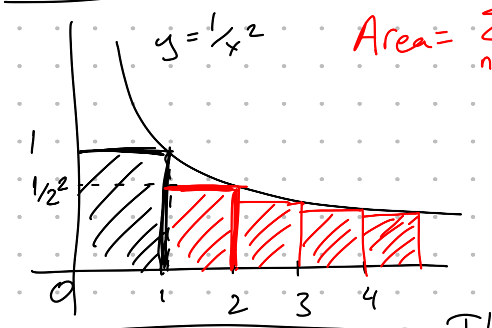
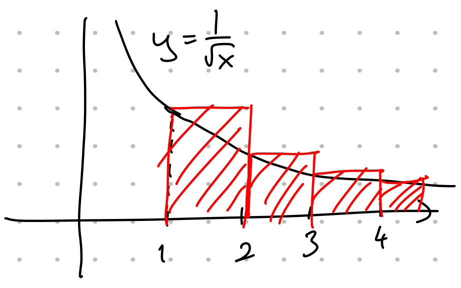
Recall:



Area $= \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}dx = 1$

In particular the sequence $\{s_n\}$ of partial sums is bounded. Note that $\{s_n\}$ is also monotonic (increasing). Thus, $\{s_n\}$ is CONV by the Monotonic Sequence theorem.

Similarly, for $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, we have



Area $= \sum_{n=1}^{t} \frac{1}{\sqrt{n}} \geq \int_1^t \frac{1}{\sqrt{x}}dx = \int_1^t \frac{1}{x^{1/2}}dx$

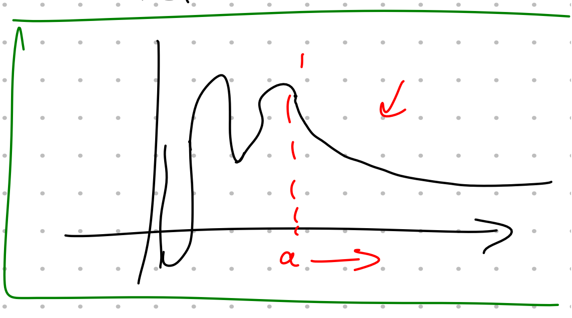
$\downarrow$ DIV $\quad \downarrow$ as $t \to \infty$

$\infty \qquad \infty$

The Integral Test Suppose f is cont., positive, decreasing on $[1,\infty)$ and let $a_n = f(n)$. Then

1) If $\int_1^{\infty} f(x)dx$ is CONV, then $\sum_{n=1}^{\infty} a_n$ is CONV

2) If $\int_1^{\infty} f(x)dx$ is DIV, then $\sum_{n=1}^{\infty} a_n$ is DIV.



Example Test the series

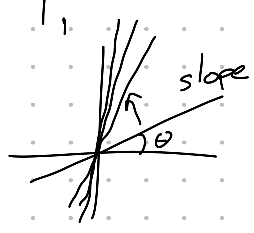
$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for CONV or DIV

$f(x) = \frac{1}{x^2+1}$ is cont., positive, decreasing on $[1,\infty)$

Thus, we can apply the Integral Test.

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}dx = \lim_{t\to\infty} \arctan x \Big|_1^t$$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$



In particular, $\int_1^{\infty} f(x)dx$ is CONV. Therefore, $\sum_{n=1}^{\infty} a_n$ is CONV.

Example For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p-series) CONV?

If $p \leq 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ DIV by the test for divergence.

If $p > 0$, then $f(x) = \frac{1}{x^p}$ cont., positive and decreasing on $[1,\infty)$

Recall $\int_1^{\infty} \frac{1}{x^p}dx$ is CONV if and only if $p > 1$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is CONV if and only if } p > 1}$$

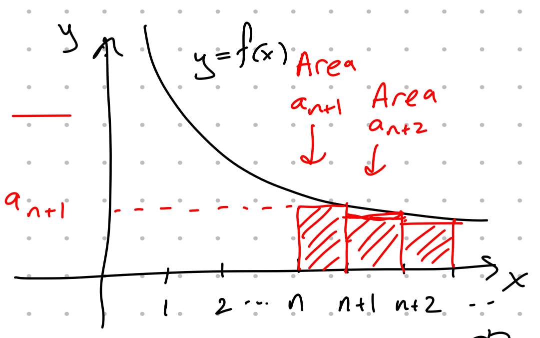
Estimating the Sum of a Series

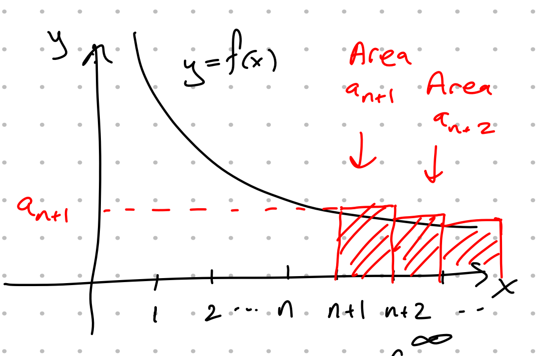
Given a CONV $\sum_{n=1}^{\infty} a_n$, how can we approximate the sum s? What is the error?

We can use the n^{th} partial sum s_n to estimate s.

$s_n = a_1 + a_2 + \cdots + a_n$.

The Remainder $= R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$





$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^{\infty} f(x)dx \qquad R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)dx$

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

Remainder Estimate for the Integral Test (f is cont., positive, decreasing on $[1,\infty)$)

Example a) Approximate the sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$ by using the sum of the first 10 terms.

b) Estimate the error involved in this approximation.

c) How many terms are required to ensure that the sum is accurate to within 0.0005?

ⓐ $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$ ← calculator

ⓑ Error $= R_n \leq \int_n^{\infty} f(x)dx$

$$\int_n^\infty \frac{1}{x^3}dx = \lim_{t\to\infty}\int_n^t x^{-3}dx = \lim_{t\to\infty} -\frac{x^{-2}}{2}\Big|_n^t$$

$$= \lim_{t\to\infty} -\frac{1}{2t^2} + \frac{1}{2n^2} = \frac{1}{2n^2}$$

for $n=10$,

$$R_{10} \le \frac{1}{2(10)^2} = \frac{1}{200} = 0.005$$

(c) We want error ≤ 0.0005. In other words we want n such that $R_n \le 0.0005$.

$$R_n \le \int_n^\infty f(x)dx = \frac{1}{2n^2} \le 0.0005$$

$a \le b$, $\frac{1}{a} \ge \frac{1}{b}$

$$\frac{1}{n^2} \le 0.001 \qquad n^2 \ge \frac{1}{0.001} = 1000$$

$$n \ge \sqrt{1000} \simeq 31.6 \qquad n \ge 32$$

$$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx$$

$S_n + R_n = S_n + (S - S_n) = S$

$$S_n + \int_{n+1}^\infty f(x)dx \le S \le S_n + \int_n^\infty f(x)dx$$

$n=10$

$$S_{10} \simeq 1.197532 \qquad \int_{11}^\infty \frac{1}{x^3}dx = \frac{1}{2(11)^2} \qquad \int_{10}^\infty \frac{1}{x^3}dx = \frac{1}{2(10)^2}$$

$$\underset{a}{1.201664} \le S \le \underset{b}{1.202532}$$

$$\frac{a+b}{2} \simeq 1.2021 \qquad \text{error} \le 0.0005$$

11.4 The Comparison Tests

The Comparison Test Suppose that Σa_n and Σb_n are series with positive terms and $a_n \le b_n$

i) If $\sum_{n=1}^\infty b_n$ is CONV, then $\sum_{n=1}^\infty a_n$ is CONV.

ii) If $\sum_{n=1}^\infty a_n$ is DIV, then $\sum_{n=1}^\infty b_n$ is DIV

Example Determine whether $\sum_{n=1}^\infty \frac{5}{2n^2+4n+3}$ is CONV or DIV.

$$\frac{5}{2n^2+4n+3} \le \frac{5}{2n^2}$$

$$\sum_{n=1}^\infty \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^\infty \frac{1}{n^2} \quad \text{(p-series with } p>1\text{) CONV}$$

By the comparison test $\sum_{n=1}^\infty \frac{5}{2n^2+4n+3}$ is also CONV.

Example Test $\sum_{k=1}^\infty \frac{\ln k}{k}$ for CONV or DIV.

$\ln k \ge 1$ for $k \ge 3$ $\qquad \frac{\ln k}{k} \ge \frac{1}{k}$ for $k \ge 3$

$\sum_{k=1}^\infty \frac{1}{k}$ is DIV (p-series with $p \le 1$) $\qquad$ Therefore $\Sigma \frac{\ln k}{k}$ is DIV by the comparison test

Think about $\Sigma \frac{1}{2^n+1}$ $\qquad \frac{1}{2^n+1} < \frac{1}{2^n}$ $\qquad \Sigma \frac{1}{2^n}$ is CONV (Geometric series $|r| = \frac{1}{2} < 1$)

Thus, $\Sigma \frac{1}{2^n+1}$ is also CONV.

However, it is much harder to use the comparison test for

$\Sigma \frac{1}{2^n-1}$ Instead we use the following test:

The Limit Comparison Test Suppose that Σa_n and Σb_n with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ is finite and $c > 0$

then either both series CONV or both DIV.

Example $\sum_{n=1}^\infty \frac{1}{2^n-1}$ $\qquad \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{\frac{1}{2^n(1-\frac{1}{2^n})}}{\frac{1}{2^n}} = \frac{\frac{1}{1-0}}{1} = 1 \ne 0$$

Since $\Sigma \frac{1}{2^n}$ is CONV (Geometric with $|r|<1$), $\sum_{n=1}^\infty \frac{1}{2^n-1}$ is also CONV by the Limit Comparison Test