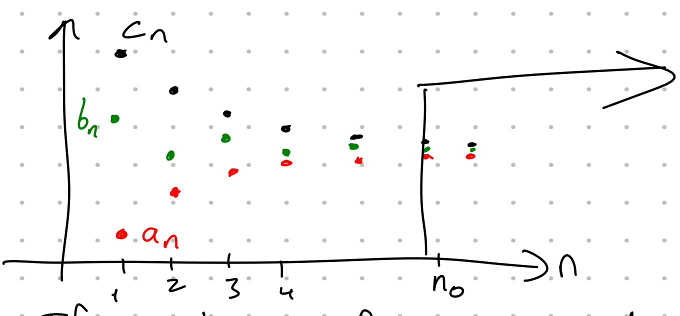
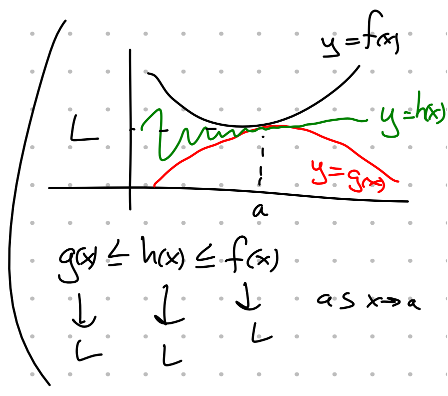
Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant,

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c a_n = c\left(\lim_{n\to\infty} a_n\right)$ • $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$ • $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ if $p>0$ and $a_n>0$

The Squeeze Theorem for Sequences





If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$.

Theorem If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

$-|a_n| \le a_n \le |a_n|$ for all n

$\downarrow$ 0, $\downarrow$ 0 as $n\to\infty$

By the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$

Example Find $\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{\frac{1}{\cancel{n}}\cancel{n}}{\frac{1}{n}(n+1)} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}$ $\searrow 0$

$= \frac{1}{1+0} = 1$

Example Calculate $\lim_{n\to\infty}\frac{\ln n}{n}$

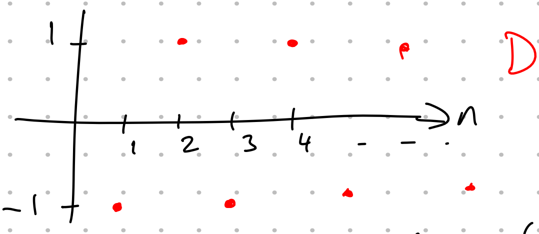
(Recall: $f(x) = \frac{\ln x}{x}$ then $a_n = \frac{\ln n}{n} = f(n)$

If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$)

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{x}\nearrow 0}{1} = 0$

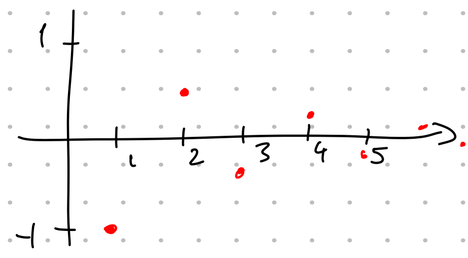
thus, $\lim_{n\to\infty}\frac{\ln n}{n} = 0$

Example Determine whether $a_n = (-1)^n$ is CONV or DIV.



DIV

Example Evaluate $\lim_{n\to\infty}\frac{(-1)^n}{n}$ if it exists.



CONV
$L=0$
$a_n \to 0$
as $n\to\infty$

$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$

Since $|a_n| \to 0$,

by the theorem above

we get $a_n \to 0$

(as $n\to\infty$).

Theorem: If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$

Example Find $\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)$.

Since sin is cont. everywhere, $\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{\pi}{n}\right)$ $\downarrow 0$

$= \sin(0) = 0$

Example Find $\lim_{n\to\infty}\frac{n!}{n^n}$ if it exists.

$a_1 = \frac{1}{1}$ $a_2 = \frac{2!}{2^2} = \frac{1\cdot 2}{2\cdot 2}$ $a_3 = \frac{1\cdot 2\cdot 3}{3\cdot 3\cdot 3}$

$a_n = \frac{1\cdot 2\cdot 3\cdot \cdots \cdot n}{n\cdot n\cdot n\cdot \cdots \cdot n} = \frac{1}{n}\underbrace{\left(\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdots\frac{n}{n}\right)}_{\le 1}$ ≤ 1 ≤ 1 ≤ 1 ≤ 1

$0 < a_n \le \frac{1}{n}$

$\downarrow$ 0, $\downarrow$ 0 as $n\to\infty$

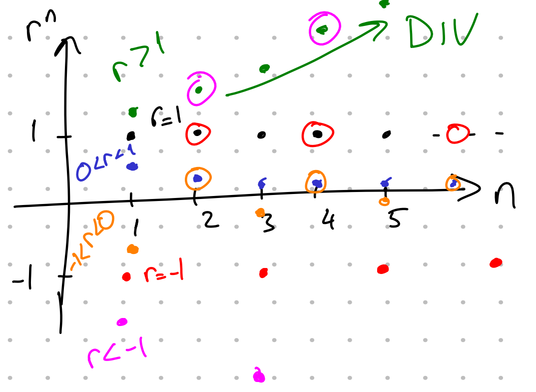
By the Squeeze Theorem, $a_n \to 0$ as $n\to\infty$

Example For what values of r is the sequence $\{r^n\}$ CONV?

(Recall: for $r=-1$ $(-1)^n$ is DIV)

If $r=0$, $a_n = r^n = 0^n = 0 \to 0$ as $n\to\infty$

If $r=1$, $1^n = 1 \to 1$ as $n\to\infty$



If $r>1$ $r^n \to \infty$ (DIV) as $n \to \infty$

If $0<r<1$ $r^n \to 0$ (CONV)

If $-1<r<0$ $r^n \to 0$ (CONV)

$\left(|r^n| = |r|^n \to 0 \right.$
$\left. \text{thus, } r^n \to 0 \right)$

• DIV If $r<-1$ r^n is DIV

The result:

$\{r^n\}$ is CONV if $-1<r\le 1$ and DIV for all other values of r and

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \end{cases}$$

Def^n A seq. $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \ge 1$, that is $a_1 < a_2 < a_3 < a_4 < \cdots$.

A seq $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \ge 1$, that is $a_1 > a_2 > a_3 > \cdots$.

A sequence is monotonic if it is either increasing or decreasing.

Example The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing.

$$a_1 = \frac{3}{1+5} = \frac{3}{6} > a_2 = \frac{3}{2+5} = \frac{3}{7} > a_3 = \frac{3}{8}$$

$$a_n = \frac{3}{n+5} > \frac{3}{n+6} = \frac{3}{(n+1)+5} = a_{n+1} \quad \text{for all } n \ge 1$$

$$0 \le a_n \le a_1 = \frac{3}{6} = \frac{1}{2} \qquad 0 \le a_n \le \frac{1}{2}$$

(bounded sequence)

Monotonic Sequence Theorem Every bounded, monotonic sequence is CONV.

e.g. $\left\{\frac{3}{n+5}\right\}$ is decreasing (so monotonic)
it is also bounded ($0 \le a_n \le \frac{1}{2}$)

So it has a limit (as a finite number)

11.2 Series

$\pi = 3.14159\ldots$

$$\pi = \underset{\underset{a_1}{\|}}{3} + \underset{\underset{a_2}{\|}}{\frac{1}{10}} + \underset{\underset{a_3}{\|}}{\frac{4}{100}} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \cdots$$

$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is called an (infinite) series and denoted by

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

Q) What is the value of $a_1+a_2+a_3+\cdots$? Does it even make sense?

e.g. $\underbrace{\frac{1}{2}}_{a_1} + \underbrace{\frac{1}{4}}_{a_2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots = 1$

$$\frac{1}{2},\ \frac{1}{2}+\frac{1}{4}=\frac{3}{4},\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8},\ \frac{15}{16},\ \frac{31}{32},\ \cdots,\ \frac{2^n-1}{2^n} = 1-\frac{1}{2^n}$$

$\frac{1}{2^n} \to 0$

partial sums and they form a new sequence $\{s_n\}$

$$\left(\begin{array}{ccccc} \text{Seq} & \longrightarrow & \text{Series} & \longrightarrow & \text{Seq} \\ \{a_n\} & & \sum_{n=1}^{\infty} a_n & & \{s_n\} \end{array} \right)$$

$s_1 = a_1 \qquad s_2 = a_1 + a_2 \qquad s_3 = a_1 + a_2 + a_3 \cdots$

Def^n Given a series $\sum_{n=1}^{\infty} a_n$ let s_n denote its nth partial sums

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

If the sequence $\{s_n\}$ is CONV and $\lim_{n\to\infty} s_n = s$ then the series $\sum_{n=1}^{\infty} a_n$ is CONV and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum. If $\{s_n\}$ is DIV then $\sum_{n=1}^{\infty} a_n$ is DIV.