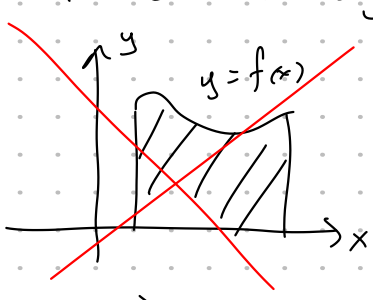
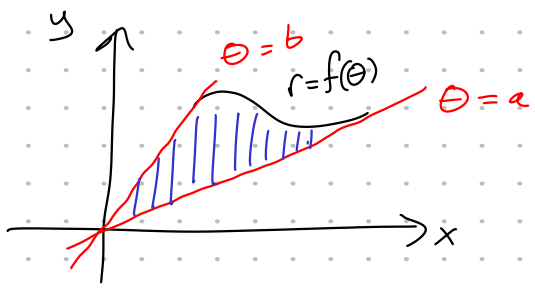
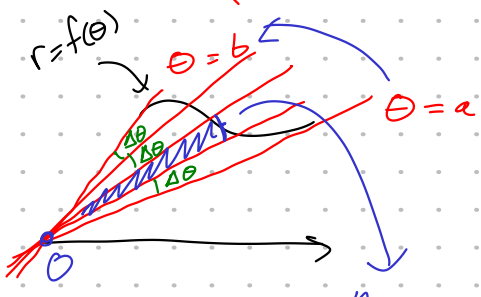
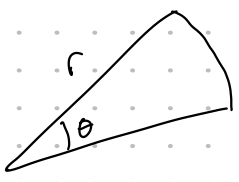
10.4 Areas and Lengths in Polar Coordinates









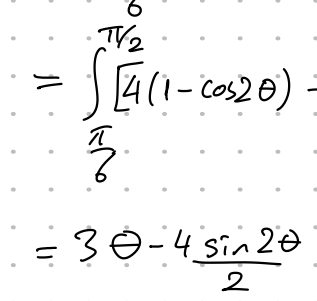
Area of a sector $= \frac{1}{2} r^2 \theta$

$$\text{Area} \approx \frac{1}{2} r^2 \Delta\theta = \frac{1}{2} (f(\theta))^2 \Delta\theta$$

$$\text{Total Area} \approx \sum_{i=1}^{n} \frac{1}{2} r^2 \Delta\theta = \sum_{i=1}^{n} \frac{1}{2} (f(\theta))^2 \Delta\theta \quad \text{in the limit}$$

$$\text{Total Area} = \int_a^b \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_a^b (f(\theta))^2 \, d\theta$$

Example Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$. (find θ by setting $r = 0$)

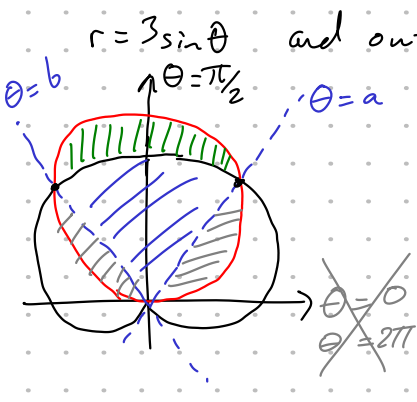


$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= 2 \left(\frac{1}{2} \int_0^{\pi/4} \cos^2 2\theta \, d\theta \right) = \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

Exercise: compute the integral using $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

Example Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardiod $r = 1 + \sin\theta$



$$A + B = \frac{1}{2} \int_a^b r^2 \, d\theta = \frac{1}{2} \int_a^b (3\sin\theta)^2 \, d\theta$$

$$B = \frac{1}{2} \int_a^b r^2 \, d\theta = \frac{1}{2} \int_a^b (1+\sin\theta)^2 \, d\theta$$

$$A = \frac{1}{2} \int_a^b (3\sin\theta)^2 \, d\theta - \frac{1}{2} \int_a^b (1+\sin\theta)^2 \, d\theta$$

$3\sin\theta = 1 + \sin\theta$
$\sin\theta = \frac{1}{2}$ (a, b)
$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$
$(0 \le \theta \le 2\pi)$

$$= \frac{1}{2} \int_{a=\pi/6}^{b=5\pi/6} \left(9\sin^2\theta - (1+\sin\theta)^2\right) d\theta$$

$$= 2 \left(\frac{1}{2} \int_a^{\pi/2} \left(9\sin^2\theta - (1+\sin\theta)^2\right) d\theta \right) = \int_{\pi/6}^{\pi/2} \left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta) \, d\theta$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$= \int_{\pi/6}^{\pi/2} \left[4(1-\cos 2\theta) - 1 - 2\sin\theta\right] d\theta = \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta) \, d\theta$$

$$= 3\theta - \frac{4\sin 2\theta}{2} + 2\cos\theta \Big|_{\pi/6}^{\pi/2} = \frac{3\pi}{2} - 2\sin\pi + 2\cos\frac{\pi}{2} - \left(\frac{3\pi}{6} - 2\sin\frac{\pi}{3} + 2\cos\frac{\pi}{6} \right)$$

($\sin\pi = 0$, $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$)

$$= \frac{3\pi}{2} - \frac{\pi}{2} = \pi$$

Arc Length

$r = f(\theta) \quad a \le \theta \le b$

$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

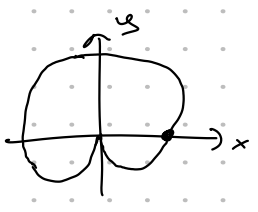
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2\frac{dr}{d\theta} r\cos\theta\sin\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2\frac{dr}{d\theta} r\cos\theta\sin\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + r^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta \qquad (r = f(\theta))$$

Example Set up an integral to find the length of the cardiod $r = 1 + \sin\theta \quad 0 \le \theta \le 2\pi$



$$L = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta = \int_0^{2\pi} \sqrt{\cos^2\theta + (1+\sin\theta)^2} \, d\theta$$

$$= \int_0^{2\pi} \sqrt{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta} \, d\theta = \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \, d\theta$$

Recall: Any point on the plane has many polar representations.

e.g. $(2, \frac{\pi}{6})$, $(-2, \frac{\pi}{6}+\pi)$, $(2, \frac{\pi}{6}+2\pi)$, ---

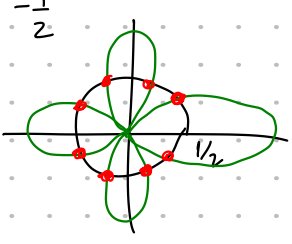
these points are the same.

Example Find all points of intersection of the curves

$r = \cos 2\theta$ and $r = \frac{1}{2}$.

Exercise: Find 4 points by setting $\cos 2\theta = \frac{1}{2}$ $\left(\begin{matrix} 0 \le \theta \le 2\pi \\ [0, 2\pi] \end{matrix}\right)$

Exercise: Find 4 more points by setting $\cos 2\theta = -\frac{1}{2}$



Chp 11 Infinite Sequences and Series

11.1 Sequences

$a_1, a_2, a_3, \cdots, a_n, \cdots$

a_1: the first term

a_n: the n^{th} term, the general term

Notation $\{a_1, a_2, a_3, \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples

a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ $\quad a_n = \frac{n}{n+1}$ $\quad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots\right\}$

b) $\left\{\frac{(-1)^n (n+1)}{3^n}\right\}$ $\quad a_n = \frac{(-1)^n (n+1)}{3^n}$ $\quad \left\{-\frac{2}{3}, \frac{3}{3^2} = \frac{1}{3}, \frac{-4}{3^3}, \cdots\right\}$

c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\quad a_n = \sqrt{n-3}$ $n \ge 3$ $\quad \{0, 1, \sqrt{2}, \sqrt{3}, \cdots\}$

Example Find a formula for the general term a_n of the sequence $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \cdots\right\}$

($a_1 = \frac{3}{5}$, $a_2 = -\frac{4}{25}$, $a_3 = \frac{5}{125}$)

$a_n = (-1)^{n-1} \frac{(n+2)}{5^n}$

Example (Fibonacci Sequence)

$\{f_n\}$ $\quad f_1 = 1$, $f_2 = 1$, $\quad f_n = f_{n-1} + f_{n-2}$ $\quad n \ge 3$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

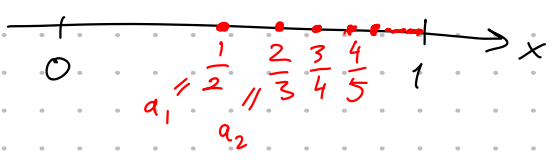
$f_4 = f_3 + f_2 = 2 + 1 = 3$ ---

$1, 1, 2, 3, 5, 8, 13, \cdots$

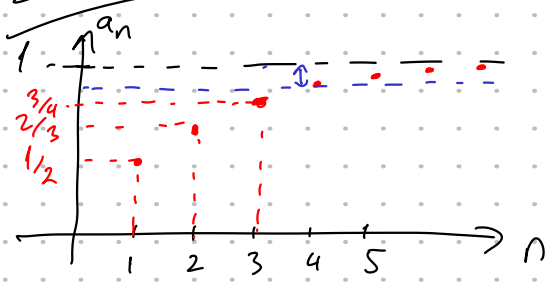
Plotting Sequences

$a_n = \frac{n}{n+1}$ $\quad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$

1st Way



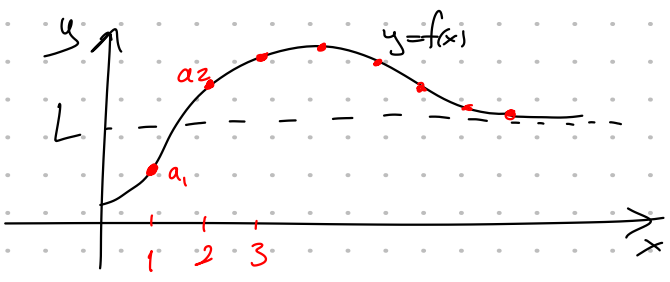
2nd Way



$\lim_{n\to\infty} \frac{n}{n+1} = 1$

In general, A sequence $\{a_n\}$ has the **limit** L and we write $\left(\lim_{n\to\infty} a_n = L\right)$ or $\left(a_n \to L \text{ as } n \to \infty\right)$ if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Theorem If $\lim_{x\to\infty} f(x) = L$, and $f(n) = a_n$ (integer) then $\lim_{n\to\infty} a_n = L$



Def^n $\lim_{n\to\infty} a_n = \infty$ means for every positive number M, there is an integer N such that if $n > N$ then $a_n > M$

e.g. $a_n = 2n + 50$

for all $n > N = 10$ $\quad a_n > 70 = M$

for all $n > N = 100$ $\quad a_n > 250 = M$

$a_n \to \infty$ as $n \to \infty$.