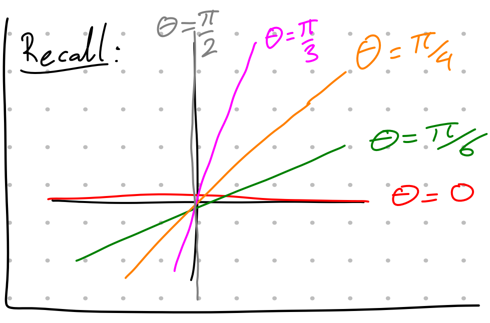
Example

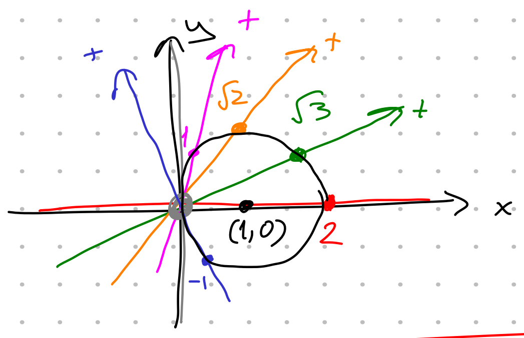
a) Sketch the curve with polar equation $r = 2\cos\theta$

b) Find a Cartesian equation for this curve.

Recall:



θ	$r=2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1



(b)

$r = 2\cos\theta$

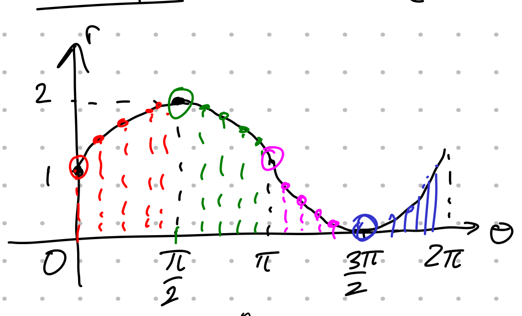
$r^2 = 2r\cos\theta$

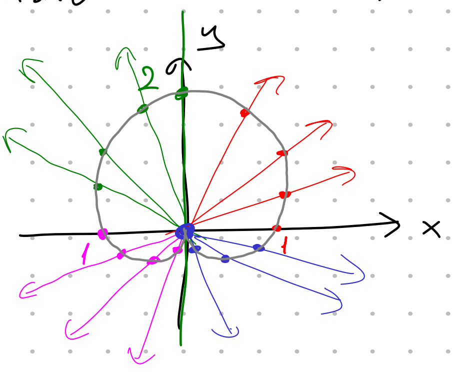
$x^2+y^2 = 2x \quad \rightarrow \quad x^2-2x+1+y^2 = +1$

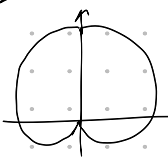
$\boxed{r^2=x^2+y^2 \quad x=r\cos\theta \quad y=r\sin\theta}$

$(x-1)^2+y^2 = 1$ the equation of a circle of radius 1 centered at (1,0)

Example Sketch the curve $r=1+\sin\theta$

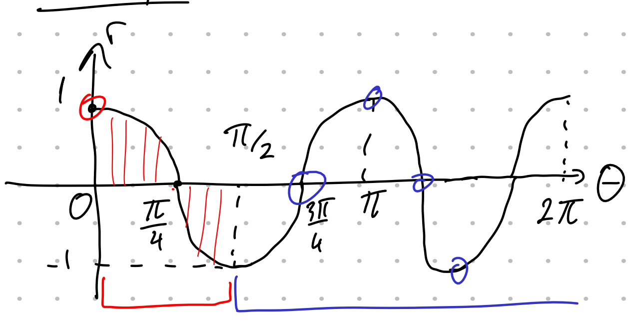


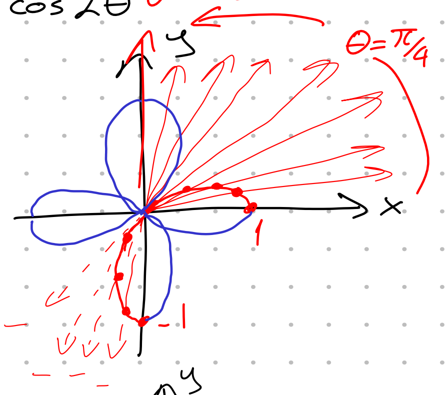


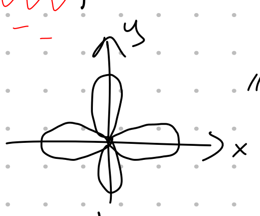


"cardioid"

Example Sketch the curve $r=\cos 2\theta$ $\theta=\pi/2$ $r<0$

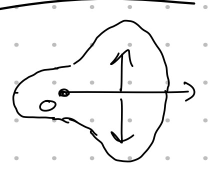
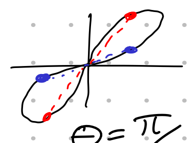
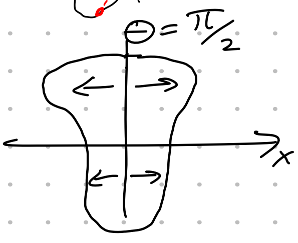






"four leaved rose"

Symmetry

If the equation does not change under	then the curve is symmetric about	picture
a) $\theta \Leftrightarrow -\theta$	the polar axis (x-axis)	
b) $r \Leftrightarrow -r$ or $\theta \Leftrightarrow \theta+\pi$	the pole (origin)	
c) $\theta \Leftrightarrow \pi-\theta$	the line $\theta=\frac{\pi}{2}$ (y-axis)	

$r=\cos(2\theta)$ "four-leaved rose"

$\cos A=\cos(-A)$

$r=\cos(2(-\theta))=\cos(-2\theta)=\cos(2\theta)$ ✓

So, symmetric about the x-axis

$r=\cos(2(\theta+\pi))=\cos(2\theta+2\pi)=\cos(2\theta)$ ✓

so, symmetric about the origin

$r=\cos(2(\pi-\theta))=\cos(2\pi-2\theta)=\cos(-2\theta)=\cos(2\theta)$

so, it is symmetric about the y-axis.

Tangents to Polar Curves

$r=f(\theta) \qquad x=r\cos\theta \qquad y=r\sin\theta$

$x=f(\theta)\cos\theta \qquad y=f(\theta)\sin\theta$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{f'(\theta)\sin\theta+f(\theta)\cos\theta}{f'(\theta)\cos\theta-f(\theta)\sin\theta}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$$

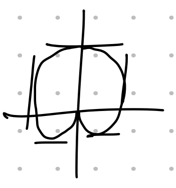
Recall that the origin corresponds to $r=0$ (θ can be anything)

When $r=0$, $\dfrac{dy}{dx}=\dfrac{\frac{df}{d\theta}\sin\theta}{\frac{df}{d\theta}\cos\theta}=\tan\theta$ if $\frac{dr}{d\theta}\neq 0$

Example a) For the cardioid $r=1+\sin\theta$, find the slope of the tangent line when $\theta=\pi/3$.

b) Find the points on the cardioid where the tangent line hor/vert.

$x=r\cos\theta \qquad y=r\sin\theta$



$x=(1+\sin\theta)\cos\theta \qquad y=(1+\sin\theta)\sin\theta$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\cos\theta\sin\theta+(1+\sin\theta)\cos\theta}{\cos\theta\cos\theta-(1+\sin\theta)\sin\theta}=\frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta-\sin\theta-\sin^2\theta}$$

$\cos^2 x = 1-\sin^2 x$

$$=\frac{\cos\theta(1+2\sin\theta)}{1-\sin\theta-2\sin^2\theta}=\frac{\cos\theta(1+2\sin\theta)}{(1-2\sin\theta)(1+\sin\theta)}$$

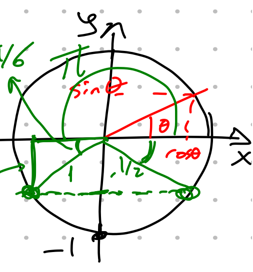
$(u=\sin\theta)$

$1-u-2u^2$

(a) $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \dfrac{\cos\left(\frac{\pi}{3}\right)\left(1+2\sin\left(\frac{\pi}{3}\right)\right)}{\left(1-2\sin\frac{\pi}{3}\right)\left(1+\sin\frac{\pi}{3}\right)} = \dfrac{\frac{1}{2}\left(1+\cancel{2}\frac{\sqrt{3}}{\cancel{2}}\right)}{\left(1-\cancel{2}\frac{\sqrt{3}}{\cancel{2}}\right)\left(1+\frac{\sqrt{3}}{2}\right)} = \dfrac{\frac{1}{2}(1+\sqrt{3})}{(1-\sqrt{3})\left(1+\frac{\sqrt{3}}{2}\right)}$

(b) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ Hor when $\dfrac{dy}{d\theta}=0$ but $\dfrac{dx}{d\theta}\neq 0$

Vert when $\dfrac{dy}{d\theta}\neq 0$ but $\dfrac{dx}{d\theta}=0$



$\dfrac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0$ $\quad \cos\theta = 0$ or $1+2\sin\theta=0$

$\frac{\pi}{2}, \boxed{\frac{3\pi}{2}}$ $\quad \sin\theta = -\frac{1}{2}$

$\pi+\frac{\pi}{6} = \frac{7\pi}{6},\ 2\pi-\frac{\pi}{6}=\frac{11\pi}{6}$

$\dfrac{dx}{d\theta} = (1-2\sin\theta)(1+\sin\theta) = 0$ $\quad 1-2\sin\theta = 0$ or $1+\sin\theta=0$

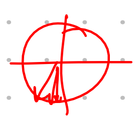
$\sin\theta = \frac{1}{2}$ $\quad \sin\theta=-1$

$\frac{\pi}{6},\ \pi-\frac{\pi}{6}=\frac{5\pi}{6}$ $\quad \boxed{\frac{3\pi}{2}}$

At $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ $\quad \frac{dy}{d\theta}=0$ but $\frac{dx}{d\theta}\neq 0$ $\quad$ Horizontal tangents

At $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad \frac{dx}{d\theta}=0$ but $\frac{dy}{d\theta}\neq 0$ $\quad$ Vertical tangents

At $\frac{3\pi}{2}$ both $\frac{dy}{d\theta} = 0 = \frac{dx}{d\theta}$ so need to compute some limits!

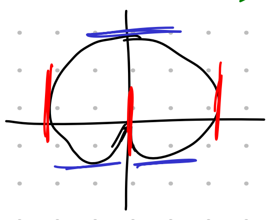


$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{dy}{dx} = \lim_{\theta\to\frac{3\pi}{2}^-} \frac{\overset{\to 0^-}{\cos\theta}\,\overset{1+2(-1)=-1}{(1+2\sin\theta)}}{\underbrace{(1-2\sin\theta)}_{1-2(-1)=3}\underbrace{(1+\sin\theta)}_{0^+}}$$

$$= \left(\lim_{\theta\to\frac{3\pi}{2}^-} \frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to\frac{3\pi}{2}^-}\frac{\overset{\to 0}{\cos\theta}}{\underset{\to 0}{1+\sin\theta}}\right) \overset{L'H}{=} -\frac{1}{3}\underbrace{\left(\lim_{\theta\to\frac{3\pi}{2}^-}\frac{\overset{-(-1)=1}{-\sin\theta}}{\underset{0^-}{\cos\theta}}\right)}_{-\infty}$$

$\frac{1+2\sin\theta}{1-2\sin\theta} \to \frac{-1}{3}$

$= \infty$



So we get a vertical tangent line at $\theta = \frac{3\pi}{2}$