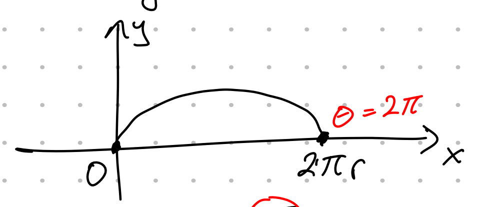
Example Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$ $\quad y = r(1-\cos\theta)$



$y = 0 \rightarrow \theta = 2\pi$

$x = 2\pi r$

$$L = \int ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\boxed{\begin{aligned} \frac{dx}{d\theta} &= r(1-\cos\theta) \\ \frac{dy}{d\theta} &= r\sin\theta \end{aligned}}$$

$$= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\; d\theta$$

$$= r\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}}\; d\theta = r\int_0^{2\pi}\sqrt{2 - 2\cos\theta}\, d\theta = \sqrt{2}\, r \int_0^{2\pi} \sqrt{1-\cos\theta}\, d\theta$$

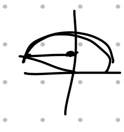
$$= \sqrt{2}\, r \int_0^{2\pi} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}\, d\theta \qquad \sqrt{a^2} = |a|$$

$$\boxed{\begin{aligned} \sin^2 x &= \tfrac{1}{2}(1 - \cos 2x) \\ \theta &= 2x \\ 2\sin^2\left(\tfrac{\theta}{2}\right) &= 1 - \cos\theta \end{aligned}}$$

$$= 2r \int_0^{2\pi} \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

$0 \le \theta \le 2\pi$

$0 \le \frac{\theta}{2} \le \pi \rightarrow \boxed{\sin\frac{\theta}{2} \ge 0}$



$$= 2r \int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= -2r\cos\left(\frac{\theta}{2}\right)\cdot 2\Big|_0^{2\pi} = -4r\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi} = -4r\cos\pi + 4r\cos 0$$

$$= -4r(-1) + 4r(1) = 8r$$

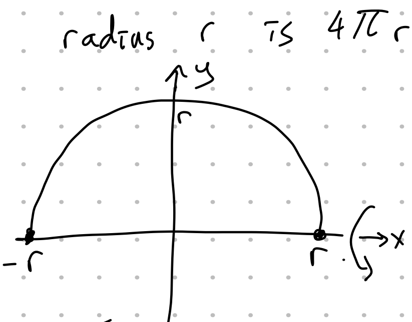
Surface Area

$$S = \int 2\pi r\, ds$$

$r = y$ if the curve is rotated about the x-axis

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Example Show that the surface area of a sphere of radius r is $4\pi r^2$



$x = r\cos\theta \qquad y = r\sin\theta \qquad 0 \le \theta \le \pi$

$$S = \int 2\pi r\, ds$$

$$\frac{dx}{d\theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$$

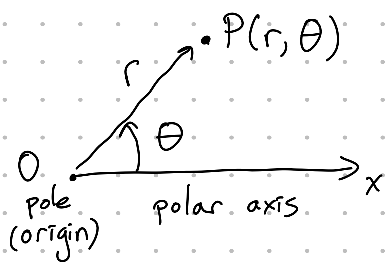
$$= \int_0^\pi 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

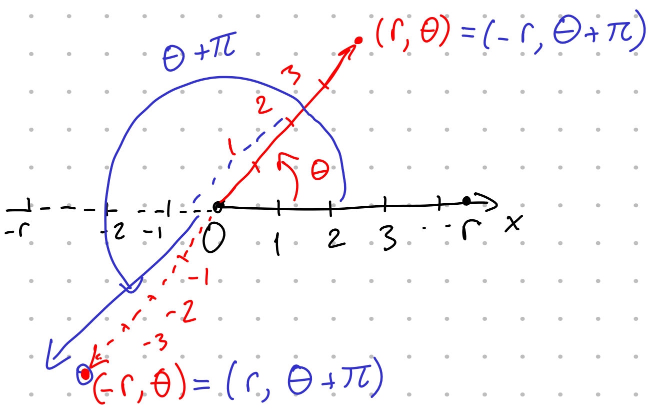
$$= 2\pi \int_0^\pi r\sin\theta \sqrt{\underbrace{r^2\sin^2\theta + r^2\cos^2\theta}_{r^2(\sin^2\theta + \cos^2\theta)}}\, d\theta = 2\pi r^2 \int_0^\pi \sin\theta\, d\theta$$

$r^2(\underbrace{\sin^2\theta + \cos^2\theta}_{1})$

$$= -2\pi r^2 \cos\theta\Big|_0^\pi = 4\pi r^2$$

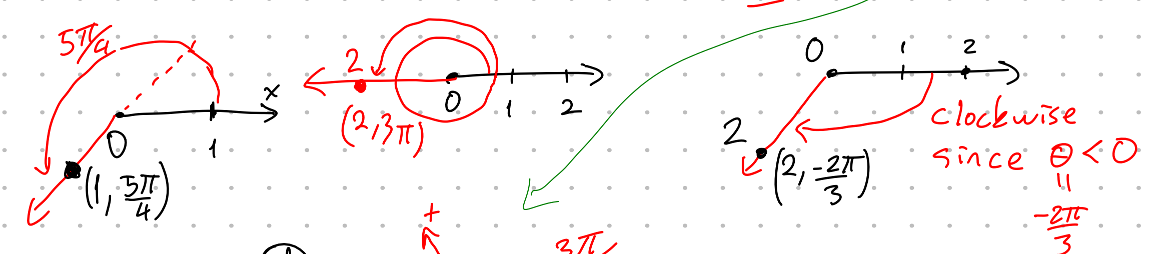
10.3 Polar Coordinates

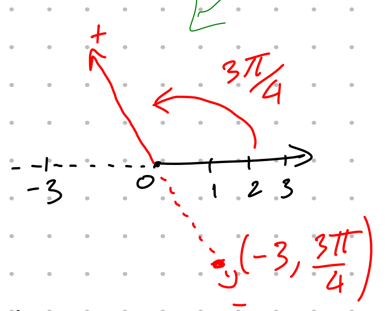


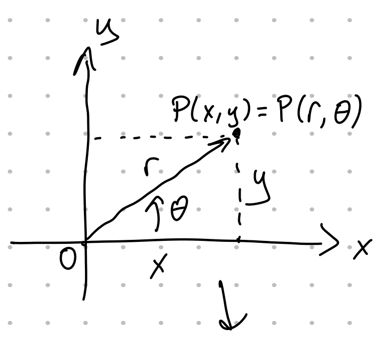


Example Plot the points whose polar coordinates are given.

a) $(1, \frac{5\pi}{4})$ $\quad$ b) $(2, 3\pi)$ $\quad$ c) $(2, -\frac{2\pi}{3})$ $\quad$ d) $(-3, 3\pi/4)$







$\rightarrow \cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$

$x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$

Example Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$r = 2 \qquad \theta = \frac{\pi}{3}$

$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$

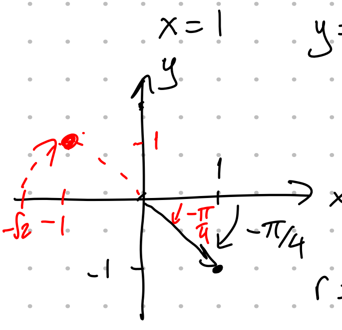
$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

Example Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

$x = 1 \qquad y = -1$

$r^2 = x^2 + y^2 = 1 + 1 = 2 \rightarrow r^2 = 2$

$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 \rightarrow \tan\theta = -1$



$\arctan(-1) = -\arctan(1) = -\frac{\pi}{4}$

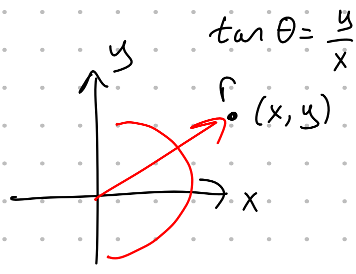
$(\sqrt{2}, -\frac{\pi}{4})$ ✓

$x=-1 \quad y=1 \qquad r^2 = x^2+y^2 = 1+1 = 2 \rightarrow r^2 = 2$

$r = -\sqrt{2},\ \theta = \frac{-\pi}{4} \qquad \tan\theta = \frac{y}{x} = \frac{1}{-1} = -1 \rightarrow \tan\theta = -1$

$\left(-\sqrt{2}, -\frac{\pi}{4}\right)$

$\tan\theta = \frac{y}{x} \rightarrow \theta = \arctan\left(\frac{y}{x}\right)$ only if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$



or if $x > 0$

then $r = +\sqrt{x^2+y^2}$

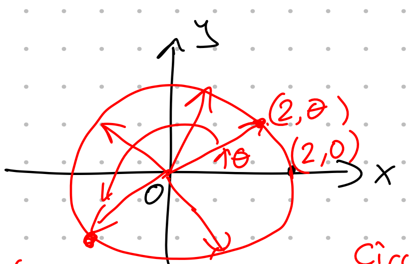
if $x < 0$ then $r = -\sqrt{x^2+y^2}$

Polar Curves

$r = f(\theta)$ $\qquad$ ($y = f(x)$ graph = all (x,y) pairs that satisfy $y=f(x)$)

Example What curve is represented by the polar equation $r = 2$?

all pairs (r,θ) of the form $(2,\theta)$

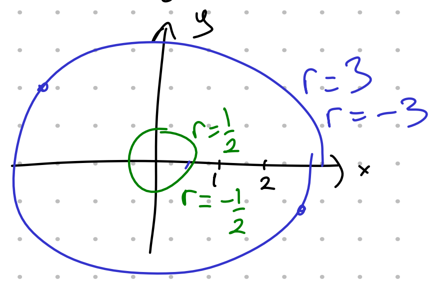


$(2, \theta+\pi) = (-2, \theta)$

Circle of radius 2.

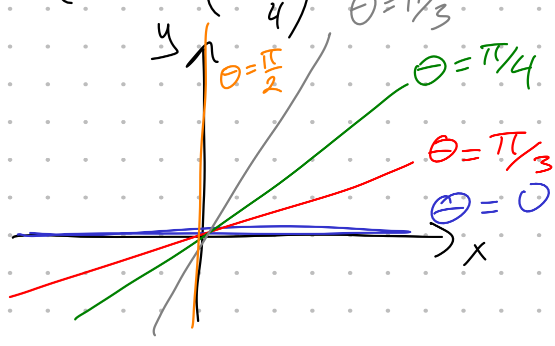
$r=2$

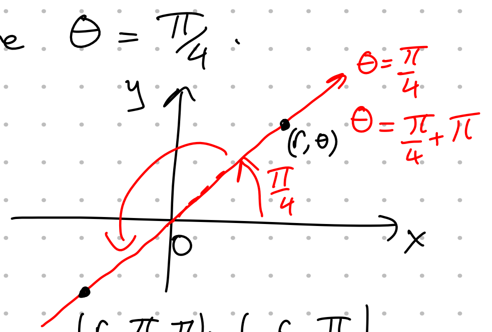
$r^2 = x^2+y^2 = 4$



Example Sketch the polar curve $\theta = \pi/4$.

$(r,\theta) \quad \left(r, \frac{\pi}{4}\right)$





$\left(r, \frac{\pi}{4}+\pi\right) = \left(-r, \frac{\pi}{4}\right)$