Example a) Find the tangent to the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1-\cos\theta)$$ at the point where $\theta = \frac{\pi}{3}$.

b) At what points is the tangent horizontal/vertical?

(a) $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{r}(\sin\theta)}{\cancel{r}(1-\cos\theta)} = \left.\frac{\sin\theta}{1-\cos\theta}\right|_{\theta=\frac{\pi}{3}}$$

$$= \frac{\sin(\pi/3)}{1-\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{1}{\cancel{2}}} = \sqrt{3} = m$$

$$x_0 = r(\theta - \sin\theta) = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$y_0 = r(1-\cos\theta) = r\left(1-\cos\frac{\pi}{3}\right) = r\left(1-\frac{1}{2}\right) = \frac{r}{2}$$

$$\boxed{y - y_0 = m(x-x_0)} \to y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

(b) $$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

Hor: $\sin\theta = 0$ but $1-\cos\theta \neq 0$

$\theta = \dots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$

$\theta = n\pi$ integer multiples of π

$1-\cos\theta = 0 \to \cos\theta = 1$

$\theta = \dots, -2\pi, 0, 2\pi, 4\pi, \dots$

$\theta = 2n\pi$ (even multiples of π)

So if θ is an odd multiple of π, $(\theta = (2n+1)\pi)$

we get horizontal tangents.

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

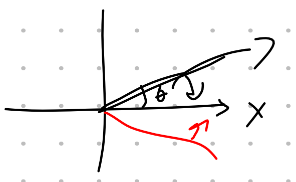
Vert: $\sin\theta \neq 0$ but $1-\cos\theta = 0$

$\theta = 2n\pi$

when $\theta = 2n\pi$, $\sin\theta$ is also 0.

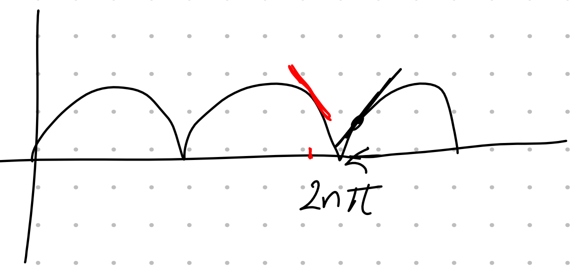
$$\lim_{\theta \to 2n\pi^{\pm}} \frac{dy}{dx} = \lim_{\theta \to 2n\pi^{\pm}} \frac{\sin\theta \to 0}{1-\cos\theta \to 0}$$

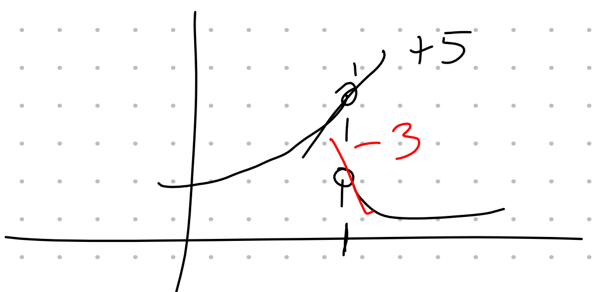
$$\overset{L'H}{=} \lim_{\theta \to 2n\pi^{\pm}} \frac{\cos\theta \to 1}{\sin\theta \to 0^{\pm}} = \pm\infty$$



(Note $\theta \to 2n\pi$ DNE)

At $\theta = 2n\pi$ we get vertical tangents.





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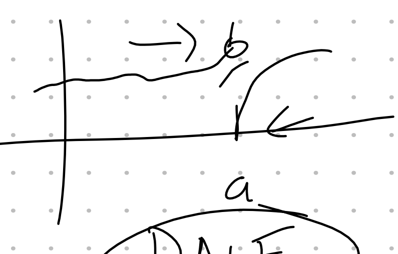

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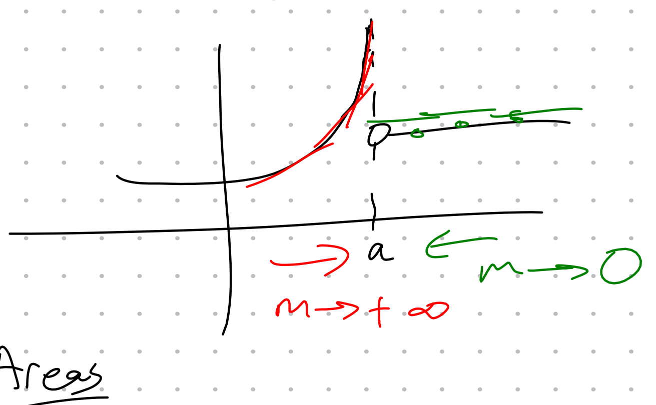
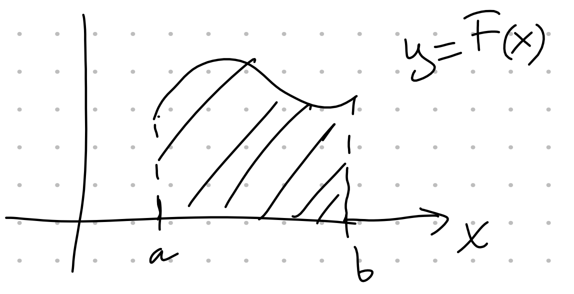


$$\lim_{t\to a} \frac{dy}{dx} = \lim_{t\to a} m$$

Areas



$$A = \int_a^b F(x)\,dx = \int_a^b y\,dx$$

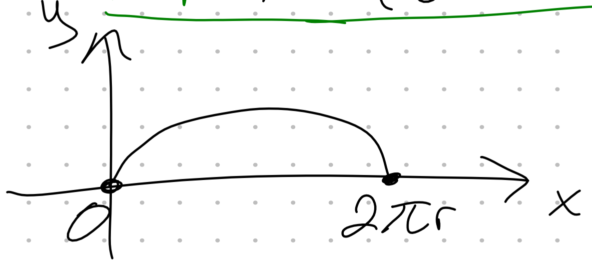
$x = f(t) \quad y = g(t) \qquad f(\beta) = b$

$dx = f'(t)\,dt \qquad f(\alpha) = a$

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt$$

Example Find the area under one arch of the cycloid.

$2\pi r = x = r(\theta - \sin\theta) \qquad y = r(1-\cos\theta)$



$$A = \int_0^{2\pi r} y\, dx$$

$$\boxed{dx = r(1-\cos\theta)\,d\theta}$$

$$= \int_0^{2\pi} r(1-\cos\theta)\, r(1-\cos\theta)\, d\theta$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2 d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= r^2(\theta - 2\sin\theta)\Big|_0^{2\pi} + r^2 \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= 2\pi r^2 + r^2 \frac{1}{2} \int_0^{2\pi} (1+\cos 2\theta)\, d\theta$$

$$= 2\pi r^2 + \frac{r^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi}$$

$$= 2\pi r^2 + \frac{r^2}{2}(2\pi) = 3\pi r^2$$

Arc Length

$$L = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \boxed{\frac{dx}{dt} = f'(t)}$$

$x = f(t) \qquad y = g(t) \qquad f(\beta) = b$

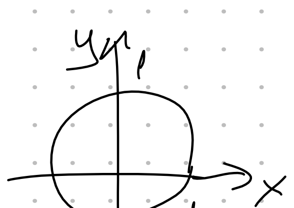
$dx = f'(t)\,dt \qquad f(\alpha) = a$

$$dx = \frac{dx}{dt}\, dt \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + \left(\frac{dy/dt}{dx/dt}\right)^2\right)}\, dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example $\quad x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$

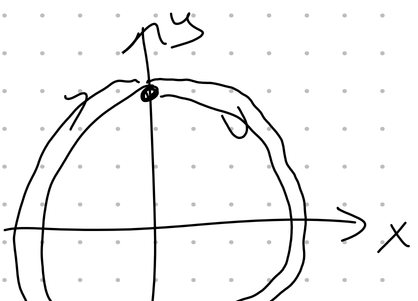


$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = \cos t$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1}}\, dt$$

$$= \int_0^{2\pi} 1\, dt = t\Big|_0^{2\pi} = 2\pi$$

$x = \sin 2t \qquad y = \cos 2t \qquad 0 \le t \le 2\pi$



$$\frac{dx}{dt} = 2\cos 2t \qquad \frac{dy}{dt} = -2\sin 2t$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\underbrace{4\cos^2 2t + 4\sin^2 2t}_{4(\cos^2 + \sin^2)}}\, dt$$

$$= \int_0^{2\pi} \sqrt{4}\, dt = \int_0^{2\pi} 2\, dt = 2t \Big|_0^{2\pi} = 4\pi$$

$$= 2(2\pi)$$

twice the circumference of the unit circle since we are traversing the unit circle twice!