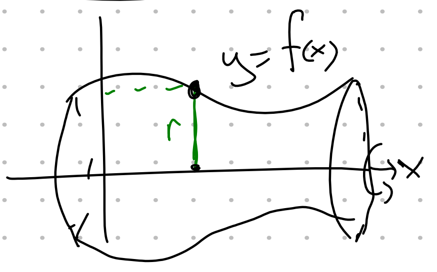
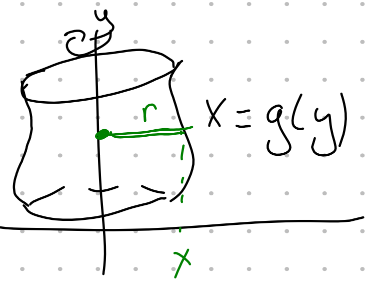
Recall: $2\pi r \ell$ $\quad \ell = ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

$$\boxed{S = \int 2\pi r\, ds}$$

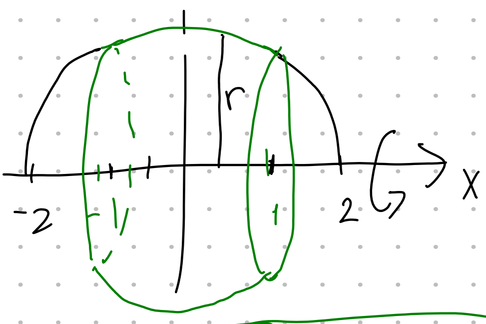


$r = y$ if the surface is obtained by revolving the curve about the x-axis



$r = x$ if the surface is obtained by revolving the curve about the y-axis

Example The curve $y = \sqrt{4-x^2}$ $\quad -1 \le x \le 1$ is rotated about the x-axis. Find the area of the resulting surface.



$$S = \int 2\pi r\, ds \qquad (r=y)$$

$$= \int_{-1}^{1} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\boxed{y = \sqrt{4-x^2} = (4-x^2)^{1/2} \qquad y' = \frac{1}{\cancel{2}}(4-x^2)^{-1/2}(-\cancel{2}x) \qquad y' = \frac{-x}{(4-x^2)^{1/2}}}$$

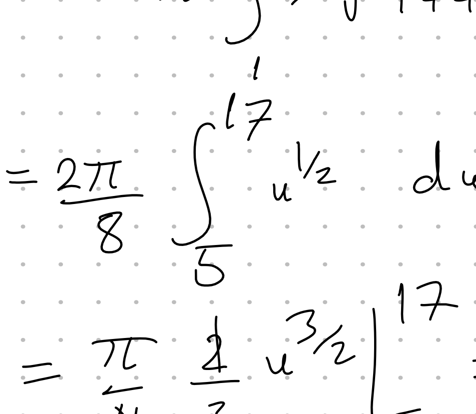
$$= 2\pi \int_{-1}^{1} (4-x^2)^{1/2}\left(1+\frac{x^2}{4-x^2}\right)^{1/2} dx$$

$$= 2\pi \int_{-1}^{1} (4-x^2)^{1/2}\left(\frac{4\cancel{-x^2}\cancel{+x^2}}{4-x^2}\right)^{1/2} dx$$

$$= 2\pi \int_{-1}^{1}\left(\cancel{(4-x^2)}\left(\frac{4}{\cancel{4-x^2}}\right)\right)^{1/2} dx = 2\pi \int_{-1}^{1} 2\,dx$$

$$= 4\pi x \Big|_{-1}^{1} = 8\pi$$

Example $y = x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface.



SOL 1: $S = \int 2\pi r\, ds$

$r = x \qquad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$y' = 2x$

$$S = \int_1^2 2\pi x \sqrt{1+(2x)^2}\,dx$$

$$= 2\pi \int_1^2 x\sqrt{1+4x^2}\,dx$$

$$\boxed{\begin{aligned} &u = 1+4x^2 \\ &du = 8x\,dx \\ &x = 2 \to u = 1+4(2)^2 = 17 \\ &x = 1 \to u = 1+4 = 5 \end{aligned}}$$

$$= \frac{2\pi}{8}\int_5^{17} u^{1/2}\,du$$

$$= \frac{\pi}{\cancel{4}_{2}}\frac{\cancel{2}}{3}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

SOL 2 $\quad ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad$ again! $r = x$ this does not change

$$S = \int 2\pi r\, ds = \int_1^4 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$\boxed{y = x^2 \to x = y^{1/2} \to x' = \frac{1}{2}y^{-1/2} = \frac{1}{2y^{1/2}}}$$

$$S = 2\pi \int_1^4 y^{1/2}\left(1+\frac{1}{4y}\right)^{1/2} dy = 2\pi \int_1^4 y^{1/2}\left(\frac{4y+1}{4y}\right)^{1/2} dy$$

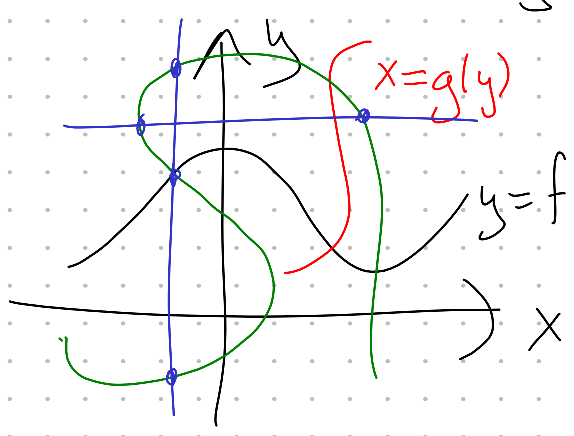
$$= \cancel{2}\pi \int_1^4 \left(\cancel{y}\left(\frac{4y+1}{\cancel{4}\cancel{y}}\right)\right)^{1/2} dy = \pi \int_1^4 (4y+1)^{1/2}\,dy$$

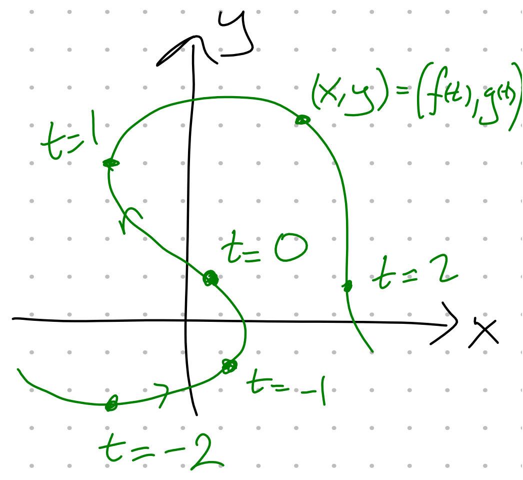
$$\boxed{\begin{aligned} &u = 4y+1 \qquad && y = 4 \to u = 16+1 = 17 \\ &du = 4\,dy && y = 1 \to u = 4+1 = 5 \end{aligned}}$$

$$= \frac{\pi}{4}\int_5^{17} u^{1/2}\,du = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

the exact same integral as in SOL1

10.1 Curves Defined by Parametric Equations





Here x and y coordinates of the particle are functions of a third variable t called the parameter and

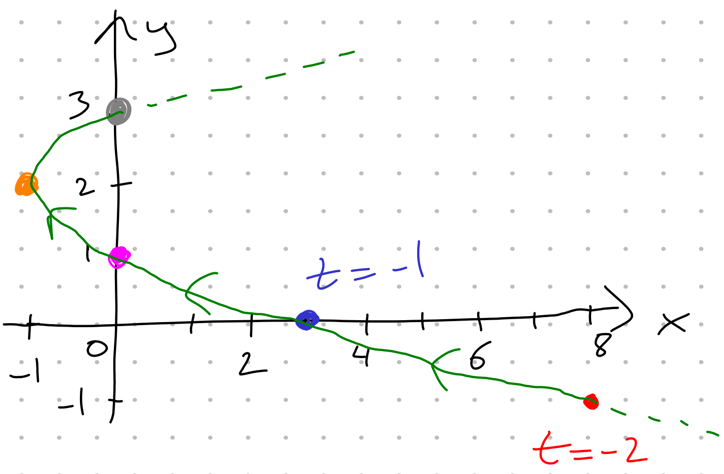
$x = f(t)$ $\quad y = g(t)$

are called the parametric equations

Example Sketch and identify the curve defined by the parametric equations

$x = t^2 - 2t \qquad y = t+1$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3



$y = t+1 \rightarrow t = y-1$

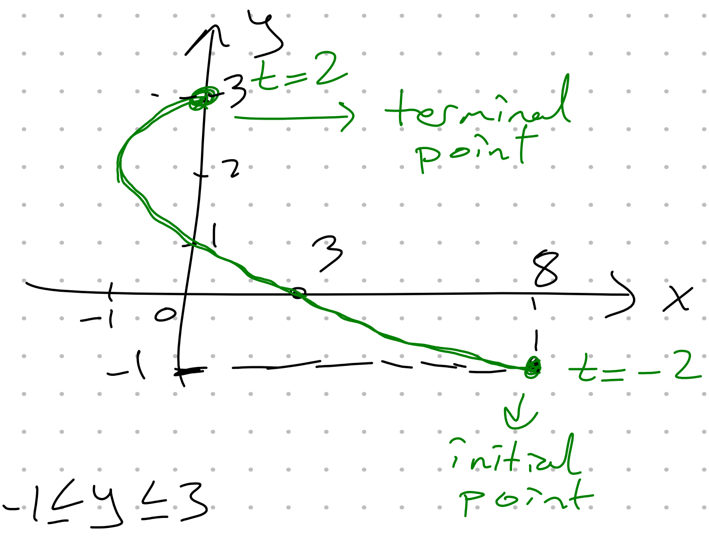
$$x = t^2 - 2t = (y-1)^2 - 2(y-1)$$
$$x = y^2 - 2y + 1 - 2y + 2$$
$$x = y^2 - 4y + 3$$

In the example above, there is no restriction on t. That's the parametric equation is the whole parabola.

$x = t^2 - 2t$

$y = t+1$

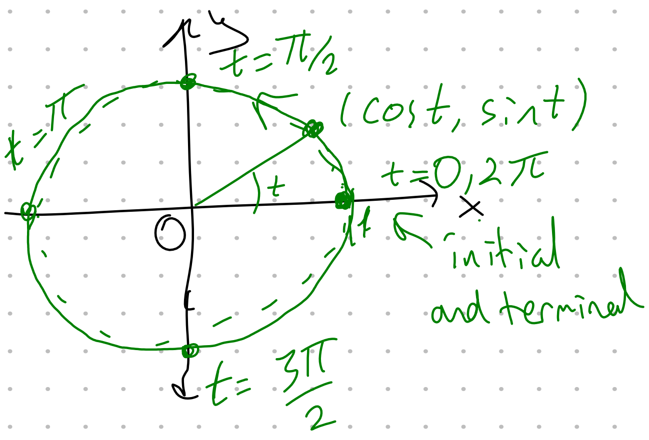
$-2 \le t \le 2$



$x = y^2 - 4y + 3 \quad -1 \le y \le 3$

Example What curve is represented by the following parametric equations?

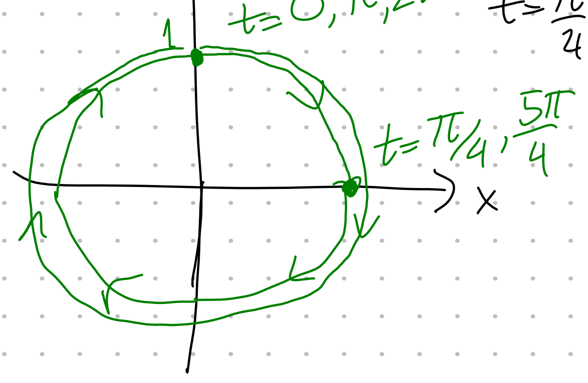
$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$



$x^2 + y^2 = \cos^2 t + \sin^2 t$

$x^2 + y^2 = 1$

Example $x = \sin 2t \quad y = \cos 2t \quad \underline{0 \le t \le 2\pi}$



$t = \frac{\pi}{4}$ $\quad x = \sin\frac{\pi}{2} = 1$

$y = \cos\frac{\pi}{2} = 0$

$x^2 + y^2 = \sin^2 2t + \cos^2 2t$

$x^2 + y^2 = 1$