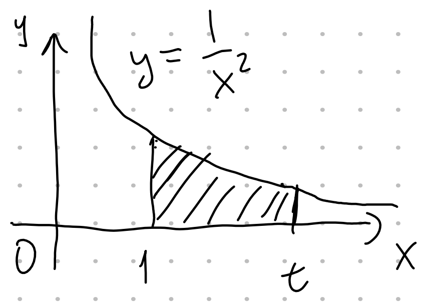
7.8 Improper Integrals

Type I: Infinite Intervals



$$A(t)=\int_1^t \frac{1}{x^2}\,dx=\int_1^t x^{-2}\,dx$$

$$=-x^{-1}\Big|_1^t=-t^{-1}-(-1)$$

$$=-\frac{1}{t}+1=\underline{1-\frac{1}{t}}$$

Thus, $\lim_{t\to\infty} A(t)=1$

$$\int_1^\infty \frac{1}{x^2}\,dx=\lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx=1$$

Defⁿ a) If $\int_a^t f(x)\,dx$ exists for every number $t\geq a$, then

$$\int_a^\infty f(x)\,dx=\lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{provided}$$

this limit exists (as a finite number)

b) If $\int_t^b f(x)\,dx$ exists for every number $t\leq b$ then

$$\int_{-\infty}^b f(x)\,dx=\lim_{t\to-\infty}\int_t^b f(x)\,dx \quad \text{provided}$$

this limit exists.

$\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called <u>convergent</u> if the corresponding limit exists and <u>divergent</u> otherwise.

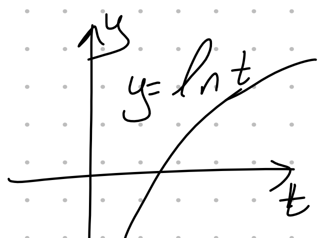
c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are <u>CONV</u> then we <u>define</u>

$$\int_{-\infty}^\infty f(x)\,dx=\int_{-\infty}^a f(x)\,dx+\int_a^\infty f(x)\,dx$$

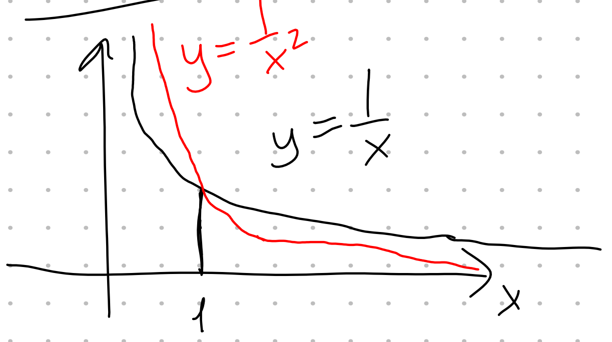
Example Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is CONV or DIV.

$$\int_1^\infty \frac{1}{x}\,dx=\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx=\lim_{t\to\infty}\ln x\Big|_1^t$$

$$=\lim_{t\to\infty}\ln t-\underbrace{\ln 1}_{=0}=\infty$$



<u>DIV</u>



Example $\int_{-\infty}^0 xe^x\,dx=\lim_{t\to-\infty}\int_t^0 xe^x\,dx$

$$=\lim_{t\to-\infty}\left(xe^x\Big|_t^0-\int_t^0 e^x\,dx\right)$$

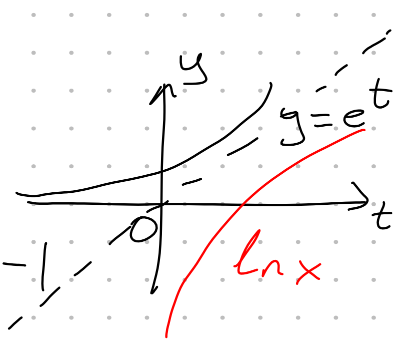
$u=x \quad dv=e^x\,dx$

$du=dx \quad v=e^x$

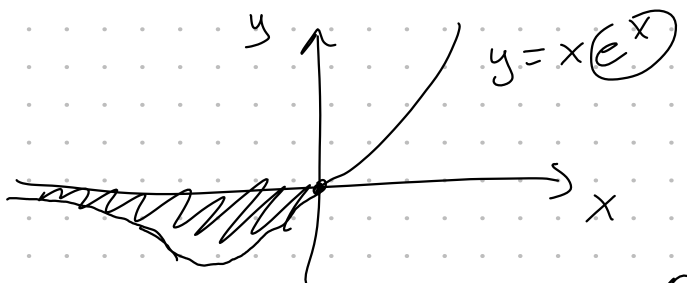
$$=\lim_{t\to-\infty}\left(0-te^t-e^x\Big|_t^0\right)$$

$$=\lim_{t\to-\infty}\left(-te^t-1+e^t\right)$$

$\downarrow \; 0 \qquad \downarrow \; -1 \qquad \downarrow \; 0$



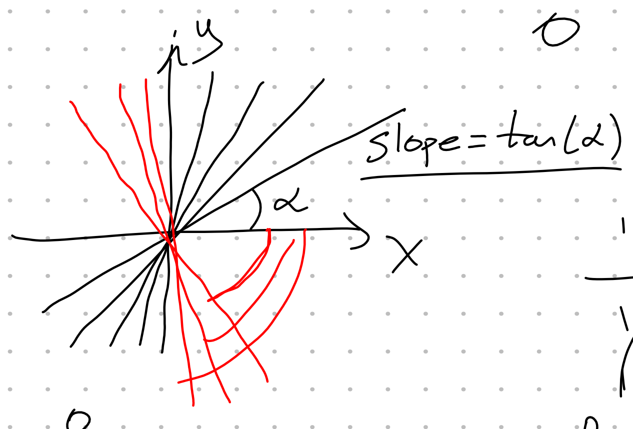
$$\lim_{t\to-\infty} te^t=\lim_{t\to-\infty}\frac{t}{e^{-t}}\overset{L'H}{=}\lim_{t\to-\infty}\frac{1}{-e^{-t}}=0$$

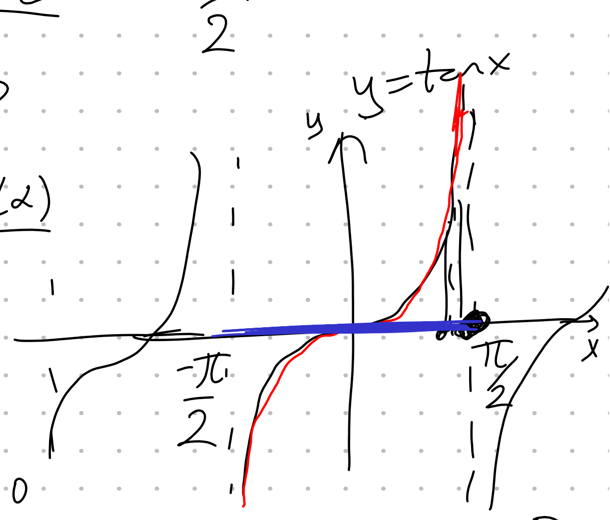


Example $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1}(x)\Big|_0^t$$

$$= \lim_{t\to\infty} \tan^{-1}(t) - \underbrace{\tan^{-1}(0)}_{=0} = \frac{\pi}{2}$$

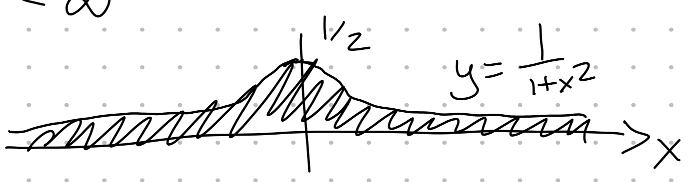




$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \tan^{-1}x\Big|_t^0$$

$$= \lim_{t\to-\infty} \underbrace{\tan^{-1}0}_{=0} - \tan^{-1}t = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$



Example For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ CONV?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big|_1^t$$

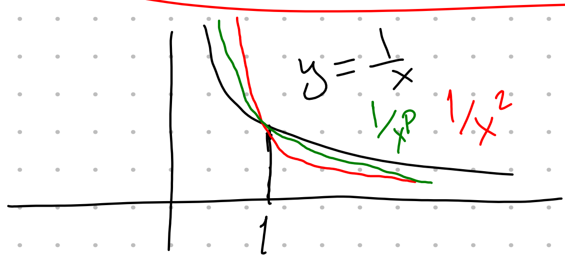
$$= \lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

If $-p+1 > 0$ then the limit is DIV

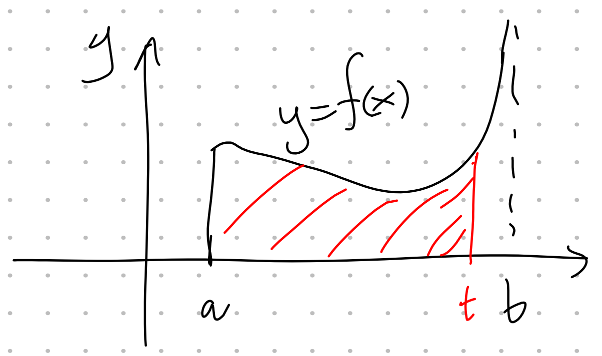
If $-p+1 < 0$ then the limit is CONV

If $1 < p$ then the integral is CONV

If $p \le 1$ then the integral is DIV



Type 2: Discontinuous Integrand



$A(t) = \int_a^t f(x)\,dx$

Def^n If f is discont. at b,

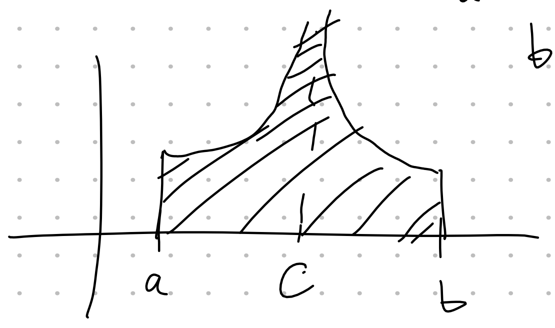
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

If f is discont. at a,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

If $a < c < b$ and f is discont. at c

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ provided both integrals are CONV.



Example $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$ Vert. asymp at 2

$$= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 (x-2)^{-1/2}\,dx$$

$\left(\begin{array}{l} u = x-2 \\ du = dx \end{array}\right)$

$$= \lim_{t\to 2^+} 2(x-2)^{1/2}\Big|_t^5$$

$$= \lim_{t\to 2^+} 2(3)^{1/2} - 2(t-2)^{1/2} = \lim_{t\to 2^+} 2\sqrt{3} - 2\underbrace{\sqrt{t-2}}_{\downarrow\, 0}$$

$= 2\sqrt{3}$ ✓ CONV.

Example $\int_0^3 \frac{1}{x-1}\,dx$ Wrong sol:

$$\cancel{= \ln|x-1|\Big|_0^3 = \ln|3-1| - \ln|0-1| = \ln 2 - \ln 1 = \ln 2}$$

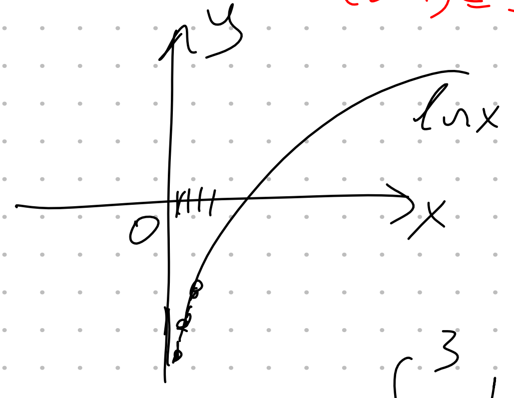
Vert. Asymp at $x=1$

$$\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$$

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \ln|x-1|\Big|_0^t$$

$$= \lim_{t\to 1^-} \ln|\underbrace{t-1}_{\text{neg.}}| - \underbrace{\ln|-1|}_{\substack{\parallel \\ 0}} = \lim_{t\to 1^-} \underbrace{\ln(\overbrace{1-t}^{\to 0^+})}_{\substack{\downarrow \\ -\infty}}$$

$-(t-1) = -t+1 = 1-t$



$\int_0^1$ is DIV

this already means $\int_0^3 \frac{1}{x-1}\,dx$ is DIV