Example $\dfrac{4x}{x^3-x^2-x+1} = \dfrac{R(x)}{Q(x)}$ $Q(1)=0 \to (x-1)$ is a factor of $Q(x)$

$Q(x) = x^2(x-1)-(x-1) = (x-1)(x^2-1)$

$= (x-1)(x-1)(x+1) = (x-1)^2(x+1)$

$$\frac{R(x)}{Q(x)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = \int\left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\right)dx$$

$$= A\int \frac{1}{x-1}dx + B\int \frac{1}{(x-1)^2}dx + C\int \frac{1}{x+1}dx$$

$u = x-1 \quad du = dx$

$$= A\ln|x-1| + B\int u^{-2}du + C\ln|x+1| + K$$

$$= A\ln|x-1| - B(x-1)^{-1} + C\ln|x+1| + K$$

$$\frac{4x}{\underbrace{x^3-x^2-x+1}_{=(x-1)^2(x+1)}} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

(roots $-1, +1$) $4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$x=1$ $4 = 2B \to B = 2$

$x=-1$ $-4 = 4C \to C = -1$

$x=0$ $0 = -A + B + C = -A + 2 - 1$

$A = 2-1 \to A = 1$

$$= A\ln|x-1| - B(x-1)^{-1} + C\ln|x+1| + K$$

$$= \ln|x-1| - 2(x-1)^{-1} - \ln|x+1| + K$$

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated

e.g. If ax^2+bx+c is a factor of $Q(x)$ $(b^2-4ac<0)$

then we will see terms of the form

$\dfrac{Ax+B}{ax^2+bx+C}$ in the partial fraction decomposition.

e.g.

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

$$\boxed{\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

Example $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}dx$ $2x^2-x+4 = R(x)$, $x^3+4x = Q(x)$ $Q(x) = x(x^2+4)$ (the only is 0)

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C)x$$

$$2x^2-x+4 = (A+B)x^2 + Cx + 4A$$

Two polynomials are equal if and only if all the coefficients are the same.

$2 = A+B \longrightarrow 2 = 1+B$

$-1 = C$ $B=1$

$4 = 4A \to A=1$

$$\int \frac{2x^2-x+4}{x(x^2+4)}dx = \int\left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx$$

$$= \ln|x| + \int \frac{x}{x^2+4}dx - \int \frac{1}{x^2+4}dx$$

$u = x^2+4 \quad du = 2x\,dx$ using the above formula

$$= \ln|x| + \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + K$$

Example $\displaystyle\int \frac{5x^2+35}{3x^2-12x+15}\,dx$ We need to use poly. long division

$$\begin{array}{r} \frac{5}{3} \\ 3x^2-12x+15 \,\big)\, 5x^2 \quad +35 \\ -\;\; 5x^2-20x+25 \\ \hline R(x) = 20x+10 \end{array}$$

$5x^2+35 = \frac{5}{3}(3x^2-12x+15) + 20x+10$

$$\frac{5x^2+35}{3x^2-12x+15} = \frac{5}{3} + \frac{20x+10}{3x^2-12x+15}$$

$b^2-4ac<0$ complete this to a square.

$\frac{5x}{3}$

$$3x^2-12x+15 = 3(x^2-4x+5)$$
$$= 3(\underline{x^2-4x+4}+1)$$
$$= 3((x-2)^2+1)$$

$$\frac{20x+10}{3x^2-12x+15} = \cancel{\frac{Ax+B}{3x^2-12x+15}}$$ already of the form we want

$$\int \frac{20x+10}{3x^2-12x+15}\,dx = \frac{10}{3}\int \frac{2x+1}{(x-2)^2+1}\,dx$$

$u = x-2 \quad du = dx$

$\downarrow$

$x = u+2$

$$= \frac{10}{3}\int \frac{2(u+2)+1}{u^2+1}\,du$$

$$= \frac{10}{3}\int \frac{2u+5}{u^2+1}\,du$$

$$= \frac{10}{3}\left(\int \frac{2u}{u^2+1}\,du + 5\int \frac{1}{u^2+1}\,du\right)$$

$v = u^2+1$

$dv = 2u\,du$

$$= \frac{10}{3}\left(\int \frac{1}{v}\,dv + 5\tan^{-1}(u)\right)$$

$$= \frac{10}{3}\left(\ln|v| + 5\tan^{-1}(u)\right) + K$$

$$= \frac{10}{3}\left(\ln|(x-2)^2+1| + 5\tan^{-1}(x-2)\right) + K$$

$+\frac{5}{3}x$

Case IV: $Q(x)$ contains repeated irreducible quadratic factors.

e.g. If $(ax^2+bx+c)^r$ is a factor of $Q(x)$ $(b^2-4ac<0)$

then

$$\frac{A_1x+B_1}{ax^2+bx+c},\ \frac{A_2x+B_2}{(ax^2+bx+c)^2},\ \ldots,\ \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

will appear in the decomposition.

Example Write out the form of the partial fraction decomposition of the function

$$\frac{R(x)}{Q(x)} = \frac{x^3+x^2+1}{x^4(x-1)(x^2+x+1)(x^2+1)^3}.$$ How many constants do you need in the decomposition?

$$\frac{R(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x-1} + \frac{Fx+G}{x^2+x+1} + \frac{Hx+I}{x^2+1}$$
$$+ \frac{Jx+K}{(x^2+1)^2} + \frac{Lx+M}{(x^2+1)^3}$$ 13 constants

7.5 Strategy for Integration

1) Simplify the integrand if possible

2) Look for an obvious substitution

3) Classify the integrand

a) Trig. int. $\int \sin^3(x)\cos^4(x)\,dx$

b) Rational fn. $\int \frac{P(x)}{Q(x)}\,dx$

c) Radicals $\sqrt{x^2-a^2}$, $\sqrt{x^2+a^2}$, $\sqrt{a^2-x^2}$

4) Try u-sub or integ. by parts

5) Repeat from Step 1

$(e^{2x} = (e^x)^2)$

Example $\int \frac{-13e^{2x}-72e^x}{e^{2x}+13e^x+36}\,dx$ $\quad u=e^x$, $du = e^x\,dx$

$= \int \frac{(-13e^x-72)\,e^x\,dx}{e^{2x}+13e^x+36} = \int \frac{-13u-72}{u^2+13u+36}\,du$

Now we have $\frac{\text{poly in } u}{\text{poly in } u}$ that is a rational function

that means we can use techniques we learned in the prev. section.