$\sqrt{a^2-x^2}$ $\quad x = a\sin t$ or $a\cos t$

$\sqrt{a^2+x^2}$ $\quad x = a\tan t$

$\sqrt{x^2-a^2}$ $\quad x = a\sec t$

Q) What about other expressions?

Example

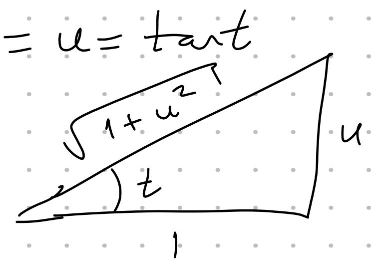
$$\int \frac{1}{\sqrt{1+(7x-8)^2}}\,dx \qquad u = 7x-8 = \tan t, \quad du = 7\,dx$$

$$= \frac{1}{7}\int \frac{1}{\sqrt{1+u^2}}\,du \qquad u = \tan t, \quad du = \sec^2 t\,dt$$

$$= \frac{1}{7}\int \frac{\sec^2 t\,dt}{\sqrt{1+\tan^2 t}} = \frac{1}{7}\int \frac{\sec^2 t}{\sqrt{\sec^2 t}}\,dt$$

$$= \frac{1}{7}\int \frac{\sec^{\cancel{2}} t}{\cancel{\sec t}}\,dt = \frac{1}{7}\int \sec t\,dt = \frac{1}{7}\ln|\underbrace{\tan t}_{u} + \sec t| + C$$

$\frac{u}{1} = u = \tan t$



$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{\sqrt{1+u^2}}} = \sqrt{1+u^2}$$

$$= \frac{1}{7}\ln\left|u + \sqrt{1+u^2}\right| + C$$

$$= \frac{1}{7}\ln\left|7x-8 + \sqrt{1-(7x-8)^2}\right| + C$$

Example $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$$3-2x-x^2 = 4-1-2x-x^2 = 4-(1+2x+x^2)$$

$$= 4-(x+1)^2 \qquad u = x+1 \quad du = dx$$

$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$2\sin t$ or $2\cos t$ $\qquad u = 2\sin t$, $u = 2\cos t$

$\sqrt{4-\cancel{\sin^2 t}}$ $\qquad \sqrt{4-4\sin^2 t} = 2\sqrt{1-\sin^2 t}$, $1-\sin^2 t = \cos^2 t$

$\cos^2 t + \sin^2 t = 1$ $\qquad = \sqrt{\cos^2 t} = \cos t$

$\sec^2 t = 1 + \tan^2 t$

7.4 Integration of Rational Functions by Partial Fractions

$$\text{Rational Function} = \frac{\text{Polynomial}}{\text{Polynomial}}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\int \frac{1}{u}\,du = \frac{1}{a}\ln|u| + C \qquad u = ax+b \quad du = a\,dx$$

$$= \frac{1}{a}\ln|ax+b| + C$$

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{2x+4-x+1}{x^2+2x-x-2}$$

$$= \frac{x+5}{x^2+x-2}$$

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \qquad \text{If } \deg P \geq \deg Q$$

we can use polynomial long division to get to the RHS where $\deg R < \deg Q$.

Example $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

$$
\begin{array}{r}
x^2+x+2 \\
x-1\,\big)\overline{\,x^3 \qquad +x} \\
-\quad x^3-x^2 \\
\hline
x^2+x \\
-\quad x^2-x \\
\hline
2x \\
-\quad 2x-2 \\
\hline
2
\end{array}
$$

$$\frac{x^3+x}{x-1} = \frac{(x-1)(x^2+x+2)+2}{x-1}$$

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

$$\int \frac{x^3+x}{x-1}\,dx$$

$$= \int\left(x^2+x+2+\frac{2}{x-1}\right)dx = \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1| + C$$

Now, we can assume $\deg R < \deg Q$ in $\frac{R(x)}{Q(x)}$. First factorize $Q(x)$ into linear and irreducible quadratic terms.

$$(ax+b), \quad (ax^2+bx+c)$$
$$b^2-4ac<0$$

then

$\frac{R(x)}{Q(x)} =$ a sum of terms of the form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^i}$

Case I: $Q(x)$ has distinct linear factors

$$Q(x) = (x-1)(x+3)(2x+3) \quad \checkmark$$
$$Q(x) = (x-1)^2(x+3) \quad \times$$
$$Q(x) = (x-1)(2x-2)$$
$$= 2(x-1)(x-1) = 2(x-1)^2 \quad \times$$

If $Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$

then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}$$

Example $\int \frac{1}{x^2-4}\,dx \qquad Q(x) = x^2-4$

$$Q(x) = \boxed{(x-2)(x+2)} = x^2-4 \quad (x=\pm 2)$$

$$\frac{1}{x^2-4} = \frac{A}{x-2}+\frac{B}{x+2}$$

$$1 = \frac{A}{\cancel{x-2}}\cancel{(x-2)}(x+2) + \frac{B}{\cancel{x+2}}(x-2)\cancel{(x+2)}$$

$$1 = A(x+2)+B(x-2)$$

$x=2$: $\quad 1 = 4A+0 \rightarrow A = \frac{1}{4}$

$x=-2$: $\quad 1 = 0+-4B \rightarrow B = \frac{-1}{4}$

$$\int \frac{1}{x^2-4}\,dx = \int\left(\frac{A}{x-2}+\frac{B}{x+2}\right)dx$$
$$= A\ln|x-2| + B\ln|x+2| + C$$
$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

Example $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$Q(x) = 2x^3+3x^2-2x = x(2x^2+3x-2)$$

$$\begin{array}{ccc} 2x & & -1 \\ x & \times & +2 \end{array}$$

$$= x(2x-1)(x+2)$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

$$\frac{x^2+2x-1}{\cancel{x(2x-1)(x+2)}} = \frac{A(2x-1)(x+2)+B(x)(x+2)+C(x)(2x-1)}{\cancel{x(2x-1)(x+2)}}$$

$$x^2+2x-1 = A(2x-1)(x+2)+B(x)(x+2)+C(x)(2x-1)$$

$x=0$

$$-1 = A(-1)(2)+0+0$$

$$-1 = -2A \qquad A = \frac{1}{2}$$

$x = \frac{1}{2}$

$$\frac{1}{4}+1-1 = 0 + B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)+0$$

$$\frac{1}{4} = \frac{5}{4}B \Rightarrow B = \frac{1}{5}$$

$x=-2$

$$4-4-1 = 0+0+C(-2)(-5)$$

$$-1 = 10C \Rightarrow C = -\frac{1}{10}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int\left(\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\right)dx$$

$$= A\ln|x| + \frac{B}{2}\ln|2x-1| + C\ln|x+2| + K$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + K$$

Case II: $Q(x)$ has repeated linear factors.

e.g. $Q(x) = (x-1)^3(x+3)^2(2x-3)$

If $(ax+b)^r$ is a factor of $Q(x)$,

$\frac{R(x)}{Q(x)}$ will include terms of the form $\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \frac{A_3}{(ax+b)^3}, \ldots, \frac{A_r}{(ax+b)^r}$

e.g.

$$\frac{R(x)}{(x-1)^3(x+3)^2(2x-3)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+3}+\frac{E}{(x+3)^2}+\frac{F}{2x-3}$$

Example

$$\int \frac{4x}{x^3-x^2-x+1}\,dx \qquad Q(x) = x^3-x^2-x+1$$

$$Q(1) = 0 \qquad (x-1) \text{ is a factor!}$$

$$Q(x) = x^2(x-1)-(x-1) = (x-1)(x^2-1)$$

$$= (x-1)(x-1)(x+1) = (x-1)^2(x+1)$$

$$\frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$$B\int (x-1)^{-2}\,dx$$