Example Calculate $\int_0^1 \tan^{-1}x\,dx$ $\quad u=\tan^{-1}x \quad dv=dx$

$du=\frac{1}{1+x^2}dx \quad v=x$

$$=\int_0^1 u\,dv = uv\Big|_0^1-\int_0^1 v\,du$$

(u-sub)

$$=x\tan^{-1}x\Big|_0^1-\int_0^1\frac{x}{1+x^2}dx$$

$u=1+x^2 \quad du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$x=0 \to u=1$

$x=1 \to u=2$

$$=\tan^{-1}(1)-0-\frac{1}{2}\int_1^2\frac{1}{u}du$$

$$=\frac{\pi}{4}-\frac{1}{2}\ln|u|\Big|_1^2=\frac{\pi}{4}-\frac{1}{2}(\ln 2-\underbrace{\ln 1}_{0})=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

7.2 Trigonometric Integrals

Example $\int\cos^3x\,dx$ $\quad u=\cos x \quad \cancel{du=-\sin x\,dx}$

there is no $\sin x\,dx$ term!

$u=\sin x \quad du=\cos x\,dx$

$$\int\underbrace{\cos^2x}_{\substack{1-\sin^2x\\1-u^2}}\,\underbrace{\cos x\,dx}_{du}$$

$$\boxed{\begin{aligned}&\cos^2x+\sin^2x=1\\ \to\ &\cos^2x=1-\sin^2x\end{aligned}}$$

$$=\int(1-\sin^2x)\cos x\,dx=\int(1-u^2)\,du=u-\frac{u^3}{3}+C$$

$$=\sin x-\frac{\sin^3x}{3}+C$$

Example $\int\sin^5x\cos^2x\,dx$ $\quad \cancel{u=\sin x \quad du=\cos x\,dx}$

$$=\cancel{\int\underbrace{\sin^5x}_{u^5}\cos x\,\underbrace{\cos x\,dx}_{du}}$$

hard to express using $\sin x$!

we try:

$u=\cos x$

$du=-\sin x\,dx$

$$=\int\underbrace{\sin^4x}_{\substack{\|\\(\sin^2x)^2=(1-\cos^2x)^2=(1-u^2)^2}}\,\underbrace{\cos^2x}_{u^2}\,\underbrace{\sin x\,dx}_{(-du)}$$

$$\boxed{\sin^2x=1-\cos^2x}$$

$$=-\int(1-u^2)^2u^2\,du=-\int(1-2u^2+u^4)u^2\,du$$

$$=-\int(u^2-2u^4+u^6)\,du=-\left(\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right)+C$$

$$=-\left(\frac{\cos^3x}{3}-\frac{2}{5}\cos^5x+\frac{\cos^7x}{7}\right)+C$$

- When evaluating $\int\cos^m(x)\sin^n(x)\,dx$

if the power of cos (m) is odd, then $u=\sin x$

and if the power of sin (n) is odd, then $u=\cos x$ subs. should work!

- When evaluating $\int\cos^{even}(x)\sin^{even}(x)\,dx$, we use the half-angle formulas:

$$\sin^2x=\frac{1}{2}(1-\cos 2x) \qquad \cos^2x=\frac{1}{2}(1+\cos 2x)$$

Example $\int_0^\pi\sin^2x\,dx=\frac{1}{2}\int_0^\pi(1-\cancel{\cos 2x})\,dx$

$$=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)\Big|_0^\pi$$

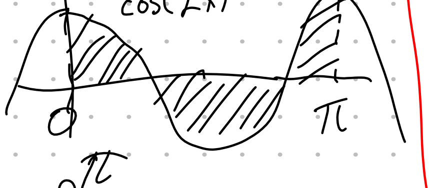
$$=\frac{1}{2}\left[\pi-\frac{\sin 2\pi}{2}-\left(0-\frac{\sin 0}{2}\right)\right]$$

$$=\frac{\pi}{2}$$

$\int\cos x\,dx=\sin x+C$

$\int\cos(5x)\,dx=\frac{\sin(5x)}{5}+C$

$u=5x \quad du=5dx$



$\int_0^\pi\cos 2x=0$

Example $\int\sin^4x\,dx=\int(\sin^2x)^2\,dx$

$$=\int\left(\frac{1}{2}(1-\cos 2x)\right)^2dx=\frac{1}{4}\int(1-2\cos 2x+\cos^2 2x)\,dx$$

$$=\frac{1}{4}\left(x-\sin(2x)\right)+\frac{1}{4}\int\cos^2 2x\,dx$$

$$\boxed{\begin{aligned}&\cos^2A=\frac{1}{2}(1+\cos 2A)\\ &\text{so }\cos^2(2x)=\frac{1}{2}(1+\cos 4x)\\ &A=2x\end{aligned}}$$

$$=\frac{x}{4}-\frac{\sin 2x}{4}+\frac{1}{4}\left(\frac{1}{2}\right)\int(1+\cos 4x)\,dx$$

$$=\frac{x}{4}-\frac{\sin 2x}{4}+\frac{1}{8}\left(x+\frac{\sin 4x}{4}\right)+C$$

$(\tan x)' = \sec^2 x \qquad (\sec x)' = \tan x \sec x \qquad \sec^2 x = 1+\tan^2 x$

Example $\int \tan^6 x \sec^4 x \, dx$

$\cancel{u = \sec x}$

$\cancel{du = \tan x \sec x \, dx}$

$$\cancel{= \int \tan^5 x \underbrace{\sec^3 x}_{u^3} \underbrace{\tan x \sec x \, dx}_{du}}$$

hard to express using $\sec x$ only

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int \underbrace{\tan^6 x}_{u^6} \underbrace{\sec^2 x}_{(1+\tan^2 x) = (1+u^2)} \underbrace{\sec^2 x \, dx}_{du}$$

$$= \int u^6 (1+u^2)\, du = \int (u^6+u^8)\, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example $\int \tan^5 t \sec^7 t \, dt$

$\cancel{u = \tan t}$

$\cancel{du = \sec^2 t \, dt}$

$$\cancel{= \int \underbrace{\tan^5 t}_{u^5} \sec^5 t \underbrace{\sec^2 t \, dt}_{du}}$$

hard to express in terms of $\tan t$

$$= \int \tan^4 t \underbrace{\sec^6 t}_{u^6} \underbrace{\tan t \sec t \, dt}_{du}$$

$u = \sec t$

$du = \tan t \sec t \, dt$

$$\tan^4 t = (\tan^2 t)^2 = (\sec^2 t - 1)^2 = (u^2-1)^2$$

$$\boxed{\begin{aligned} \sec^2 x &= 1 + \tan^2 x \\ \sec^2 x - 1 &= \tan^2 x \end{aligned}}$$

$$= \int (u^2-1)^2 u^6 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du = \int (u^{10} - 2u^8 + u^6)\, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C = \frac{\sec^{11} t}{11} - \frac{2}{9}\sec^9 t + \frac{\sec^7 t}{7} + C$$

- When evaluating $\int \tan^{\text{any}}(x) \sec^{\text{even}}(x)\, dx$

  we use $u = \tan x \quad du = \sec^2 x \, dx$

- When evaluating $\int \tan^{\text{odd}}(x) \sec^{\text{any}}(x)\, dx$

  we use $u = \sec x \quad du = \tan x \sec x \, dx$

- When evaluating $\int \tan^{\text{even}}(x) \sec^{\text{odd}}(x)\, dx$

  No short answer.