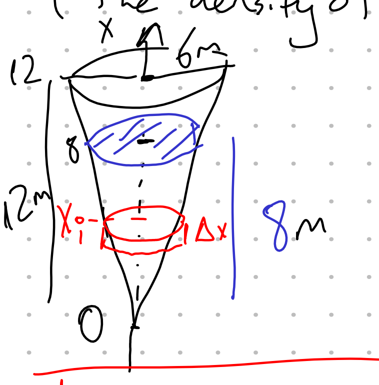
Example A tank has the shape of an inverted circular cone with height 12m and base radius 6m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is $1000\,kg/m^3$)

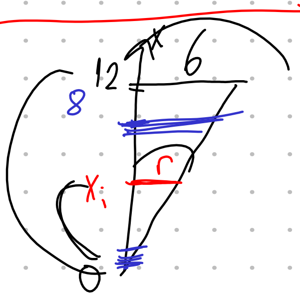
$g = 9.8\ m/s^2$



$$W_i = \text{Force} \times \text{distance}$$
$$= \text{Weight} \cdot (12 - x_i)$$
$$= m \cdot g \cdot (12 - x_i)$$
$$= \text{Volume} \cdot \text{density} \cdot g\,(12 - x_i)$$
$$= \pi r^2 h\,(1000)(9.8)(12 - x_i)$$
$$= \pi r^2 \Delta x \;\; 9800\,(12 - x_i)$$
$$= \boxed{\pi \frac{x_i^2}{4} \Delta x \;\; 9800\,(12 - x_i)} \quad W_i$$

density = $\dfrac{\text{mass}}{\text{Volume}}$

$\hookrightarrow$ mass = density $\cdot$ Vol



$$\frac{r}{x_i} = \frac{6}{\cancel{12}_2} \rightarrow r = \frac{x_i}{2}$$

$$\text{Total Work} = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9800\pi}{4} x_i^2 (12 - x_i)\Delta x$$

$$= \int_0^8 \frac{9800\pi}{4} x^2 (12 - x)\,dx$$

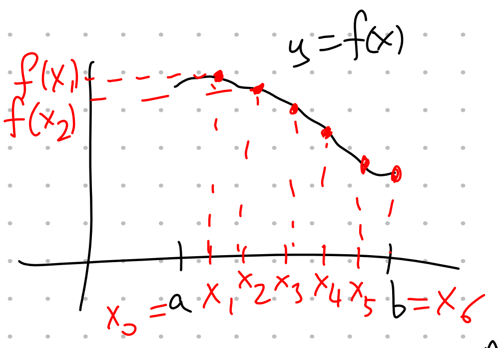
$$= \frac{9800\pi}{4} \int_0^8 (12x^2 - x^3)\,dx = \frac{9800\pi}{4}\left(\frac{\cancel{12}^{4} x^3}{\cancel{3}} - \frac{x^4}{4}\right)\Big|_0^8$$

$$= \frac{9800\pi}{4}\left(4(8)^3 - \frac{(8)^4}{4}\right) J$$

6.5 Average Value of a Function

$1, 5 \qquad \frac{1+5}{2} = 3 \qquad\qquad 1, 2, 11 \qquad \frac{1+2+11}{3} = \frac{14}{3}$

$$a_1, a_2, \ldots a_n \quad \text{average} = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\sum_{i=1}^{n} a_i}{n}$$



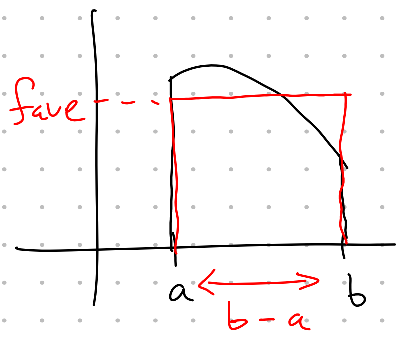
$$f_{ave} \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_6)}{6}$$
$$= \frac{\sum_{i=1}^{6} f(x_i)}{6}$$

$$f_{ave} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i)}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i)}{\frac{b-a}{\Delta x}}$$

$$\Delta x = \frac{b-a}{n} \rightarrow n = \frac{b-a}{\Delta x}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f(x_i)\Delta x}{b-a} = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



$$(b-a)\, f_{ave} = \int_a^b f(x)\,dx$$

$\downarrow$ base $\quad \downarrow$ height

Example Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

$$f_{ave} = \frac{1}{2-(-1)} \int_{-1}^{2} f(x)\,dx = \frac{1}{3}\int_{-1}^{2} (1 + x^2)\,dx$$

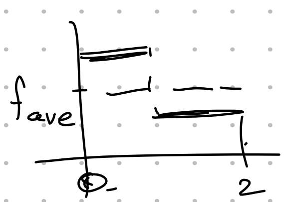
$$= \frac{1}{3}\left(x + \frac{x^3}{3}\right)\Big|_{-1}^{2} = \frac{1}{3}\left(2 + \frac{8}{3} - \left(-1 - \frac{1}{3}\right)\right)$$

$$= 2$$

The Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$



e.g. (by the prev. example) $f_{ave} = 2$
for $f(x) = 1+x^2$ on $[-1,2]$

By the MVT for integrals, there is a number c in $[-1,2]$ s.t. $f(c) = 2$

$$1+c^2 = 2 \Rightarrow c^2 = 1 \quad c = \pm 1$$

Example Find the numbers b such that the average value of $f(x) = 2+6x-3x^2$ on the interval $[0,b]$ is equal to 4.

$$4 = f_{ave} = \frac{1}{b}\int_0^b (2+6x-3x^2)\,dx$$

$$4 = \frac{1}{b}\left(2x + \frac{\cancel{6}^{3}x^2}{\cancel{2}} - \frac{\cancel{3}x^3}{\cancel{3}}\right)\Big|_0^b = \frac{1}{b}(2b+3b^2-b^3)$$

$$4 = \underbrace{2+3b-b^2} \rightarrow b^2-3b+2=0$$
$$(b-2)(b-1) = 0$$
$$b = 1, 2$$

7.1 Integration by Parts

Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$\int (f(x)g(x))'\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$
$\parallel$ by FTC
$f(x)g(x)$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by parts. $\quad u = f(x) \quad v = g(x)$

In Leibniz notation: $\quad du = f'(x)dx \quad dv = g'(x)dx$

$$\int u\,dv = uv - \int v\,du$$

Example $\int x\sin x\,dx \qquad u = x \quad dv = \sin x\,dx$
$\qquad du = dx \quad v = -\cos x$

$= \int u\,dv = uv - \int v\,du$

$= -x\cos x - \int(-\cos x)\,dx = -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + C$

Example $\int \ln x\,dx \qquad u = \ln x \quad dv = dx$
$\qquad du = \frac{1}{x}dx \quad v = x$

$= \int u\,dv = uv - \int v\,du$

$= x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx = x\ln x - \int dx = x\ln x - x + C$

Example $\int t^2 e^t\,dt \qquad u = t^2 \quad dv = e^t dt$
$\qquad du = 2t\,dt \quad v = e^t$

$\int u\,dv = uv - \int v\,du$

$= t^2e^t - \int e^t 2t\,dt = t^2e^t - 2\int te^t\,dt$

We need to apply integration by parts again!

$\qquad u = t \quad dv = e^t dt$
$\qquad du = dt \quad v = e^t$

$= t^2e^t - 2\left(uv - \int v\,du\right)$

$= t^2e^t - 2\left(te^t - \int e^t\,dt\right) = t^2e^t - 2te^t + 2e^t + C$

Example $\int e^x \sin x\,dx \qquad u = \sin x \quad dv = e^x dx$
$\qquad du = \cos x\,dx \quad v = e^x$

$= uv - \int v\,du$

$= e^x\sin x - \int e^x\cos x\,dx \qquad u = \cos x \quad dv = e^x dx$
$\qquad du = -\sin x\,dx \quad v = e^x$

$$\stackrel{!}{=} e^x \sin x - \left(uv - \int v\,du\right)$$

$$= e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x)\,dx\right)$$

So far:

$$\underbrace{\int e^x \sin x\,dx}_{I} = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x\,dx}_{I}$$

$$I = e^x(\sin x - \cos x) - I$$

$$2I = e^x(\sin x - \cos x)$$

$$I = \frac{e^x(\sin x - \cos x)}{2} + C$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

Examples next time.