Linear systems of first order diff. eq.s

Announcement: This is our last session. However, you need to attend (or watch it later) one more session of one of the other proffessors.

Example

$$x_1' = x_1 + 2x_2$$
$$x_2' = 2x_1 - 2x_2$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad X' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - 2x_2 \end{pmatrix}$$

$$\underline{X'} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} X}$$

let $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ Say λ is an eig. value and v is eig. vect. corresponding to λ then $\underline{X = e^{\lambda t} v}$ is a solution of the system.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = -2 + 2\lambda - \lambda + \lambda^2 - 4 = \lambda^2 + \lambda - 6$$

$$= (\lambda + 3)(\lambda - 2) = 0 \qquad \lambda = 2, -3 \text{ are eigenvalues.}$$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow v_1 - 2v_2 = 0 \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

set $v_2 = 1$, $\underline{v_1 = 2}$

$\boxed{v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ is an eig. vector } \lambda = 2.}$

$\lambda = -3$

$$A - \lambda I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow 2v_1 + v_2 = 0$$

set $v_2 = 1$ $\quad v_1 = -\frac{1}{2}$

$\boxed{v = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \text{ is an eig. vect. corr. to } \lambda = -3}$

$$X = c_1 e^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

Example

$x_1(0) = 5$, $x_2(0) = 1$

$$x_1' = 2x_1 + x_2$$
$$x_2' = -3x_1 - 2x_2$$

$$\rightsquigarrow X' = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} X$$

(where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $X(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$)

$$A = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 3 = \lambda^2 - 1 = 0$$

$$= (\lambda - 1)(\lambda + 1) = 0 \qquad \lambda = \pm 1$$

$\lambda = -1$

$$A - \lambda I = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \qquad 3v_1 + v_2 = 0$$

$\underline{\text{set } v_2 = 3}$ $\rightarrow v_1 = -1$ sothat there are no fractions in v_1

$$v = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad v_1 + v_2 = 0 \qquad \text{set } v_2 = 1,\ v_1 = -1$$

$$v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$X = c_1 e^{-t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -c_1 - c_2 \\ 3c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & -1 & 5 \\ 3 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -5 \\ 0 & -2 & 16 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -5 \\ 0 & 1 & -8 \end{array}\right) \rightarrow c_2 = 8$$

$c_1 + c_2 = -5$

$c_1 = 3$

$$\underline{X = 3e^{-t}\begin{pmatrix} -1 \\ 3 \end{pmatrix} - 8e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}} = \underline{\begin{pmatrix} -3e^{-t} + 8e^{t} \\ 9e^{-t} - 8e^{t} \end{pmatrix}}$$

Example

$$x' = Ax \qquad A = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \left|\begin{array}{ccc} -\lambda & 2 & -3 \\ -2 & 4-\lambda & -3 \\ -2 & 2 & -1-\lambda \end{array}\right| \begin{array}{cc} -\lambda & 2 \\ -2 & 4-\lambda \\ -2 & 2 \end{array}$$

$$= (+\lambda)(4-\lambda)(+1+\lambda) + 12 + 12 - [6(4-\lambda) + 6\lambda + 4(+1+\lambda)]$$

$$= \lambda(4-\lambda)(1+\lambda) + \cancel{24} - [\cancel{24} - \cancel{6\lambda} + \cancel{6\lambda} + 4 + 4\lambda]$$

$$= \lambda(4-\lambda)\underline{(1+\lambda)} - 4\underline{(1+\lambda)} = (1+\lambda)(4\lambda - \lambda^2 - 4)$$

$$= -(\lambda+1)(\lambda^2 - 4\lambda + 4) = -(\lambda+1)(\lambda-2)^2 \qquad \lambda = -1,\ \text{mult.} = 1$$

$\lambda = 2$, mult. 2

$\lambda = -1$

$$A - \lambda I = \begin{pmatrix} 1 & 2 & -3 \\ -2 & 5 & -3 \\ -2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 9 & -9 \\ 0 & 6 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \begin{array}{l} v_1 + 2v_2 - 3v_3 = 0 \\ v_2 - v_3 = 0 \end{array}$$

Set $v_3 = 1$

$\rightarrow v_2 = 1$

$v_1 + 2 - 3 = 0 \qquad v_1 = 1$

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda = -1$$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} -2 & 2 & -3 \\ -2 & 2 & -3 \\ -2 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad 2v_1 - 2v_2 + 3v_3 = 0$$

v_2 and v_3 are both free. first set $\underline{v_2 = 1 \text{ and } v_3 = 0}$

then $v_1 = 1$ so $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eig. vect. corr. to $\lambda = 2$.

Next set $v_2 = 0$ and $v_3 = 1$ then $v_1 = -\frac{3}{2}$

$v = \begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix}$ is another eig. vect. corr. to $\lambda = 2$.

(It is clear that these eig. vectors are linearly indep.t)

$$X = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{2t}\begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix}$$

Q Why did we choose $v_2 = 1, v_3 = 0$ and $v_2 = 0, v_3 = 1$?

$$E_2 = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;\middle|\; 2v_1 - 2v_2 + 3v_3 = 0 \right\}$$

set $v_2 = t$, $v_3 = s$

$$= \left\{ \begin{pmatrix} v_1 \\ t \\ s \end{pmatrix} \;\middle|\; \cancel{2}v_1 = \frac{2t - 3s}{2} \right\} = \left\{ \begin{pmatrix} t - \frac{3s}{2} \\ t \\ s \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}$$

$$\begin{pmatrix} t - \frac{3}{2}s \\ t \\ s \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2}s \\ 0 \\ s \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix}$$

Example $x' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} x$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \rightsquigarrow \det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

$\lambda = \pm 2i$ $\quad e^{\lambda t} v$

$\lambda = 2i$

$$A - \lambda I = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \sim \begin{pmatrix} 2 & 2i \\ -2 & -2i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$$

$v_1 + i v_2 = 0$ Set $v_2 = 1 \rightsquigarrow v_1 = -i$

$v = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. (Because we have complex conj. pair of eig. values we don't have to consider $\lambda = -2i$)

$$e^{\lambda t} v = e^{2it}\begin{pmatrix} -i \\ 1 \end{pmatrix} = (\cos(2t) + i\sin(2t))\begin{pmatrix} -i \\ 1 \end{pmatrix} = (\cos 2t + i\sin 2t)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 0 \\ \cos 2t \end{pmatrix} + i\begin{pmatrix} -\cos 2t \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ \sin 2t \end{pmatrix} + \begin{pmatrix} +\sin 2t \\ 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}}_{\text{Real part}} + i\underbrace{\begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}}_{\text{Imag Part.}}$$

$$X = c_1\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} + c_2\begin{pmatrix} -\cos 2t \\ \sin 2t \end{pmatrix}$$

(No explicit complex number here.)

$e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$

$e^{(a-ib)t} = e^{at}(\cos bt - i\sin bt)$

span the same space as $e^{at}\cos bt,\ e^{at}\sin bt$

$$c_1 e^{a+ib\,t} + c_2 e^{a-ib\,t} = d_1 e^{at}\cos bt + d_2 e^{at}\sin bt$$