Before class discussion.

$$\begin{pmatrix}1\\4\end{pmatrix} \rightsquigarrow 3 \qquad \begin{pmatrix}1\\-3\end{pmatrix} \rightsquigarrow -2$$

$$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \qquad Av = \lambda v$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix} = 3\begin{pmatrix}1\\4\end{pmatrix} \rightsquigarrow \begin{pmatrix}a+4b\\ c+4d\end{pmatrix} = \begin{pmatrix}3\\12\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\-3\end{pmatrix} = -2\begin{pmatrix}1\\-3\end{pmatrix} \rightsquigarrow \begin{pmatrix}a-3b\\ c-3d\end{pmatrix} = \begin{pmatrix}-2\\6\end{pmatrix}$$

$a+4b=3$
$a-3b=-2$

$c+4d=12$
$c-3d=6$

Beginning of the lecture

e) $(D-5)^{17}(D^2+4D+5)^3 y = 0$

Recall from last time: $(D-5)y=0$ $\quad y_1 = e^{5x} \quad y = c_1 y_1 = c_1 e^{5x}$

$(D^2+4D+5)y=0 \rightarrow y_1 = e^{-2x}\cos x \quad y_2 = e^{-2x}\sin x$

$y = c_1 y_1 + c_2 y_2 = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x.$

$y_1 = e^{5x} \quad y_2 = x e^{5x} \quad y_3 = x^2 e^{5x} \ \dots \quad y_{17} = x^{16} e^{5x}$

$y_{18} = e^{-2x}\cos x \qquad y_{19} = e^{-2x}\sin x$

$y_{20} = x e^{-2x}\cos x \qquad y_{21} = x e^{-2x}\sin x$

$y_{22} = x^2 e^{-2x}\cos x \qquad y_{23} = x^2 e^{-2x}\sin x$

$y_{gen} = c_1 y_1 + c_2 y_2 + \dots + c_{23} y_{23} = c_1 e^{5x} + c_2 x e^{5x} + \dots + c_{23} x^2 e^{-2x}\sin x.$

Inhomogeneous Lin. Diff. Eq. with constant coeff.

Recall $(D-k)y=0 \quad \longleftrightarrow \quad y = c_1 e^{kx}$

$(D-k)^r y = 0 \quad \Longleftrightarrow \quad y = c_1 e^{kx} + \dots + c_r x^{r-1} e^{kx}$

$(D-(a+ib))(D-(a-ib))y = 0 \Longleftrightarrow y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$

$\Vert\ (D^2 - 2aD + a^2+b^2)y = 0$

$[(D-(a+ib))(D-(a-ib))]^r y = 0 \Longleftrightarrow y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$

$\Vert\ (D^2-2aD+a^2+b^2)^r y = 0$

$+ c_3 x e^{ax}\cos bx + c_4 x e^{ax}\sin bx$

$+ \dots$

$\vdots$

$+ c_{2r-1} x^{r-1} e^{ax}\cos bx + c_{2r} x^{r-1} e^{ax}\sin bx$

Example Determine the general solution to

(1) $(D+3)(D-3)y = 10e^{2x}$ (Inhom. eq.)

Note $(D-2)e^{2x} = 0$ so if we apply $(D-2)$ to both sides, we get

(2) $(D-2)(D+3)(D-3)y = (D-2)10e^{2x} = 0$

If $z = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{3x}$, then z solves this equation (2)

However, not every solution of (2) is a solution of (1).

So in other words y is of the form of z but the coefficients for y are not arbitrary.

Note that $c_2 e^{-3x} + c_3 e^{3x}$ is the general solution for $(D+3)(D-3)y=0$ so when we are looking for a particular solution to (1), we can ignore these terms and just use $y_p = c_1 e^{2x}$. Now plug in $y_p = c_1 e^{2x}$ in (1) to figure out what c_1 is.

$y_p' = 2c_1 e^{2x}$
$y_p'' = 4c_1 e^{2x}$

(1) $(D+3)(D-3)y_p = 10e^{2x}$

$(D^2-9)y_p = y_p'' - 9y_p = 10e^{2x}$

$4c_1 e^{2x} - 9c_1 e^{2x} = 10e^{2x}$

$-5c_1 \cancel{e^{2x}} = 10\cancel{e^{2x}} \qquad c_1 = -2$

So $y_p = -2e^{2x}$ is a particular solution. So the general solution for the Inhom. (1) will be of the form

$$y_{gen} = y_p + y_{hom} = -2e^{2x} + c_1 e^{-3x} + c_2 e^{3x}$$

the general solution of the corresponding homogeneous eq. $(D+3)(D-3)y=0$

Example Show that $A(D) = D^2+4$ annihilates $F(x) = 5\cos 2x$.

that is show $A(D)F = 0$.

$A(D)F = F'' + 4F \qquad F' = -10\sin 2x$

$= -20\cos 2x + 4(5\cos 2x) \qquad F'' = -20\cos 2x$

$= 0$ ✓

Example $(D-4)(D+1)y = \cancel{15e^{4x}}\ 0$ Find the general solution.

$(D-4)e^{4x} = 0$ so $(D-4)^2(D+1)y = 0$

and y must be of the form $\underbrace{c_1 e^{-x} + c_2 e^{4x}}_{\text{general solution for the homogeneous eq.}} + \underbrace{c_3 x e^{4x}}_{y_p}$

$y_p = c_3 x e^{4x}$

$y_p' = c_3 e^{4x} + 4c_3 x e^{4x} = c_3 e^{4x}(1+4x)$

$y_p'' = 4c_3 e^{4x} + 4c_3 e^{4x} + 16c_3 x e^{4x} = 4c_3 e^{4x}(1+1+4x) = 8c_3 e^{4x}(1+2x)$

$15e^{4x} = (D-4)(D+1)y_p = (D^2-3D-4)y_p = y_p'' - 3y_p' - 4y_p = 15e^{4x}$

$$8c_3 \cancel{e^{4x}}(1+2x) - 3c_3\cancel{e^{4x}}(1+4x) - 4c_3 x \cancel{e^{4x}} = 15\cancel{e^{4x}}$$

$$(8c_3 + \cancel{16c_3x}) + (-3c_3 \; \cancel{-12c_3x}) + (0 \; \cancel{-4c_3x}) = 15 + 0x$$

$$5c_3 = 15 \qquad \underline{\underline{c_3 = 3.}} \qquad y_p = 3xe^{4x}$$

$$y_{gen} = y_{hom} + y_p = c_1e^{-x} + c_2e^{4x} + 3xe^{4x}.$$

<u>Example</u> $(D^2+1)y = 3\cos x + 4\sin x.$

$$\cos x = e^{\overset{a}{0}x}\cos(\overset{b}{1}x) \Longleftrightarrow (D^2 - 2aD + a^2+b^2)y = 0$$

$$\text{or } (D^2+1)\cos x = 0$$

similarly $\sin x = e^{0x}\sin(1x) \Longleftrightarrow (D^2+1)\sin x = 0$

So

$$\underline{(D^2+1)^2 y} = (D^2+1)(3\cos x + 4\sin x) = \underline{0}$$

y_p must be of the form $y_p = \underbrace{c_1\cos x + c_2\sin x}_{\text{is a solution to the corr. hom. eq. } (D^2+1)y=0} + \underbrace{c_3x\cos x + c_4 x\sin x}_{y_p}$

$$y_p = \boxed{c_3 x\cos x + c_4 x\sin x}$$

$$y_p' = c_3\cos x - c_3x\sin x + c_4\sin x + c_4x\cos x = \cos x(c_3 + c_4x) + \sin x(c_4 - c_3x)$$

$$y_p'' = -\sin x(c_3 + c_4x) + \cos x(c_4) + \cos x(c_4 - c_3x) + \sin x(-c_3)$$

$$= \boxed{2c_4\cos x - 2c_3\sin x - c_3x\cos x - c_4x\sin x}$$

$$(D^2+1)y_p = y_p'' + y_p = 3\cos x + 4\sin x$$

$$2c_4\cos x - 2c_3\sin x + \cancel{c_3x\cos x} - \cancel{c_3x\cos x} + \cancel{c_4x\sin x} - \cancel{c_4x\sin x}$$

$$= 3\cos x + 4\sin x$$

$$2c_4 = 3 \qquad c_4 = \frac{3}{2}$$

$$-2c_3 = 4 \qquad c_3 = -2$$

$$y_p = -2x\cos x + \frac{3}{2}x\sin x;$$

$$y_{gen} = c_1\cos x + c_2\sin x - 2x\cos x + \frac{3}{2}x\sin x.$$