Before Class discussion.

WW 11 Problem 17 is removed

3×3 real matrix

only 1 real eigenvalue $\Rightarrow$ 2 complex conjugate

(so there are 3 distinct eigenvalues and the matrix has to be non-defective)

$A_{n\times n}$ is non-defective $\Leftrightarrow$ $\mathbb{C}^n$ has a basis consisting of eigenvectors of A

Fact 1) for eigenvalue λ of multiplicity m_λ,
$$1 \le \dim(E_\lambda) \le m_\lambda$$

2) Sum of multiplicities of eigenvalues of $A_{n\times n}$ is n.
$$m_{\lambda_1} + m_{\lambda_2} + \cdots + m_{\lambda_k} = n.$$

3) this definition is equivalent to saying that
$$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \cdots + \dim(E_{\lambda_k}) = n$$
$\Longleftrightarrow$ A is non-defective.

Beginning of the Lecture

Last time $\quad y''-2y'+15y=0 \qquad$ Hint: $y=e^{rx}$

$y'=re^{rx} \quad y''=r^2e^{rx}$ $\quad r^2\cancel{e^{rx}}-2r\cancel{e^{rx}}+15\cancel{e^{rx}}=0$

$(r^2-2r+15)=0$

$(r-5)(r+3)=0$

$r=5,-3$

$y=e^{5x}$, $y=e^{-3x}$ are solutions.

So a general solution is of the form $y_{gen}=c_1e^{5x}+c_2e^{-3x}$

Example Find all solutions to $y''+4y=0$ Hint: $y=e^{rx}$

$y'=re^{rx} \quad y''=r^2e^{rx}$

$r^2\cancel{e^{rx}}+4\cancel{e^{rx}}=0$

$r^2+4=0$

$r^2=-4$

$r=\pm 2i$

$y_1=e^{2ix} = \cos(2x)+i\sin(2x)$ Complex solutions

$y_2=e^{-2ix} = \cos(2x)-i\sin(2x)$ to the diff. eq.

(So technically $y_{gen}=c_1y_1+c_2y_2 = c_1(\cos 2x+i\sin 2x)+c_2(\cos 2x-i\sin 2x)$

but we typically want real valued solutions.

Short way of getting real solutions is to use the real part and the imaginary part of y_1.

So $y_1=\underbrace{\cos 2x}_{\text{Real}}+i\underbrace{\sin 2x}_{\text{imaginary}}$ $\qquad$ So $y_{gen}=c_1\cos 2x+c_2\sin 2x$

Example a) Show that $y_1=\cos 2x$, $y_2=3(1-2\sin^2 x)$ are solutions to $y''+4y=0$

$y_1'=-2\sin 2x \qquad y_1''=-4\cos 2x$

$y_1''+4y_1=-4\cos 2x+4\cos x=0$ ✓

$y_2'=3(-4\sin x\cos x)=-12\sin x\cos x$

$y_2''=-12(\cos^2 x-\sin^2 x)$

$y_2''+4y_2=-12(\cos^2x-\sin^2x)+12(1-2\sin^2x)$

$=12\left[-\cos^2x+\cancel{\sin^2x}+1-\cancel{2}\sin^2x\right]$

$=12\left[1-\cos^2x-\sin^2x\right]=0 \qquad$ since $\cos^2x+\sin^2x=1$

b) Check if y_1 & y_2 are linearly independent.

Recall: For solutions of a (higher) homogeneous linear diff. eq., the Wronskian test is definite. (If $W[f_1\ldots f_k]$ is 0 at a point $f_1\ldots f_k$ are LD and if it is not 0, they are LI)

$$W[y_1,y_2](x)=\begin{vmatrix}\cos 2x & 3(1-2\sin^2x)\\ -2\sin 2x & -12\sin x\cos x\end{vmatrix}$$ Now, we can choose $x=0$

$$W[y_1,y_2](0)=\begin{vmatrix}1 & 3\\ 0 & 0\end{vmatrix}=0$$ So they are LD.

Fact If L is a linear diff. operator and y_p is a particular solution to $Ly=F(x)$. Then general solution to $Ly=F(x)$ is given by y_p+y_{hom} where y_{hom} is the general solution for the corresponding homogeneous eq. $(Ly=0)$

Example $y''-2y'-15y=18e^{6x}$. Verify $y_p=2e^{6x}$ is a particular solution. Find the general solution.

$y=2x$, $\cancel{-2x+y=0}$ $\qquad$ $y=2x+1$, $\cancel{-2x+y=1}$

$y_p'=12e^{6x} \qquad y_p''=72e^{6x}$

$72e^{6x}-2(12e^{6x})-15(2e^{6x})=e^{6x}(72-24-30)$

$=e^{6x}(18)=18e^{6x}$ ✓

Now we have to find the general solution to $y''-2y'-15y=0$

but this is the example from last time $y_{hom}=c_1e^{5x}+c_2e^{-3x}$

$y_{gen}=y_p+y_{hom}=2e^{6x}+c_1e^{5x}+c_2e^{-3x}$.

Example (Constant Coefficient Homogeneous Lin. Diff. Eq.)

Find the general solution to the following equations:

a) $y' - 5y = 0 \rightsquigarrow y = Ce^{5x}$

Notation: $Dy = y'$, $D^2y = y''$

$Dy - 5y = 0$

$(D-5)y = 0$

$P(D) = D - 5 \rightsquigarrow P(r) = r - 5 = 0 \to r = 5 \to \underline{\underline{y_1 = e^{5x}}}$

$y_{gen} = c_1 y_1 = c_1 e^{5x}$

b) $(D-5)^{17} y = 0$

$$\left((D-5)^2 y = (D-5)(D-5)y = (D^2 - 10D + 25)y = y'' - 10y' + 25y = 0\right)$$

$P(D) = (D-5)^{17}$

$P(r) = (r-5)^{17} = 0 \to r = 5$ with multiplicity 17.

$y_1 = e^{5x}$ $y_2 = xy_1 = xe^{5x}$ $y_3 = xy_2 = x^2e^{5x}$

$y_4 = x^3e^{5x}$ $y_5 = x^4e^{5x}$ .... $y_{17} = x^{16}e^{5x}$

$y_{gen} = c_1y_1 + c_2y_2 + \dots + c_{17}y_{17} = c_1e^{5x} + c_2xe^{5x} + c_3x^2e^{5x} + \dots + c_{17}x^{16}e^{5x}$

c) $y''' - 5y'' = 0 \to P(D) = D^3 - 5D^2 = D^2(D-5)$

$(D^3 - 5D^2)y = 0$

$P(r) = r^2(r-5) = 0$

$r = 5$ with multiplicity 1 $\to y_1 = e^{5x}$

$r = 0$ // // 2 $\to y_2 = e^{0x}$ $y_3 = xe^{0x}$

$y_2 = 1$ $y_3 = x$

$y_{gen} = c_1y_1 + c_2y_2 + c_3y_3 = c_1e^{5x} + c_2 + c_3x.$

d) $y'' + 4y' + 5 = 0$

$(D^2 + 4D + 5)y = 0$

$P(D) = D^2 + 4D + 5 \to P(r) = r^2 + 4r + 5 = (r+2)^2 + 1 = 0$

$(r+2)^2 = -1$ $r + 2 = \pm i$

$r = -2 \pm i$ $r = a \pm ib$

$y = e^{rx} = e^{(-2+i)x} = e^{-2x}e^{ix} = e^{-2x}(\cos x + i\sin x)$

$= \underbrace{e^{-2x}\cos x}_{y_1} + i\,\underbrace{e^{-2x}\sin x}_{y_2}$

$y_1 = e^{-2x}\cos x$ $y_1 = e^{ax}\cos(bx)$

$y_2 = e^{-2x}\sin x$ $y_2 = e^{ax}\sin(bx)$

$y_{gen} = c_1y_1 + c_2y_2 = \underline{c_1}e^{-2x}\cos x + \underline{c_2}e^{-2x}\sin x$