MT2 1pm

Q4.2 $V = P_4(\mathbb{R})$ $W = \{ p \in V \mid p(1)p(2) = 2p(3) \}$

Say $p \in W$ then $p(1)p(2) = 2p(3)$

but $(5p)$ $(5p)(1)(5p)(2) \overset{?}{=} 2(5p)(3)$

$25\, p(1)p(2) \overset{?}{=} 10\, p(3)$

If $p(1)p(2) = 2p(3) \neq 0$ then we may get a counter-example.

let's see if there is 1^{st} order polynomial (for simplicity).

$p(x) = a + bx$ $p(1) = a + b$ $p(2) = a + 2b$ $p(3) = a + 3b$

$0 \neq (a+b)(a+2b) = 2(a+3b)$

$a^2 + 3ab + 2b^2 = 2a + 6b$ set $b = 1$

$a^2 + 3a + 2 = 2a + 6$

$0 = a^2 + a - 4 = (a + \frac{1}{2})^2 - \frac{1}{4} - 4$ $(a + \frac{1}{2})^2 = 4 + \frac{1}{4} = \frac{17}{4}$

$a + \frac{1}{2} = \frac{\sqrt{17}}{2}$ $a = \frac{\sqrt{17} - 1}{2}$ so now

since $p(1)p(2) = 2p(3) \neq 0$

$25\, p(1)p(2) = 25(2p(3)) \neq 5(2p(3))$ so W

is not closed under scalar multiplication.

Q6. $V = M_{3\times 2}(\mathbb{R})$ $W = \{ A \in V \mid a_{11} + a_{22} = 3a_{12} + 2a_{32} \}$

Find a spanning set.

$$W = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \middle| \; \underset{\text{bound variable}}{a_{11}} = -a_{22} + 3a_{12} + 2a_{32} \right\}$$

free vars.

Actually everything except a_{11} is a free var.

$$\begin{pmatrix} -a_{22} + 3a_{12} + 2a_{32} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = a_{12}\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$+ a_{31}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{32}\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So $\left\{ \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a spanning set for W.

Q3. Given that $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\begin{vmatrix} 2a+d & 2b+e & 2c+f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{vmatrix}$

Recall that ① det is linear in each one of the rows (as long as other rows are fixed) ② $kR_i + R_j \to R_j$ does not change the det ③ $R_i \leftrightarrow R_j$ changes the sign of the det.

$$= \begin{vmatrix} 2a & 2b & 2c \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{vmatrix} + \begin{vmatrix} d & e & f \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{vmatrix} \; -2R_1+R_2 \to R_2$$

$$= 2\begin{vmatrix} a & b & c \\ 2d+g & 2e+h & 2f+i \\ 2g+a & 2h+b & 2i+c \end{vmatrix} + \begin{vmatrix} d & e & f \\ g & h & i \\ 2g+a & 2h+b & 2i+c \end{vmatrix} \; -2R_2+R_3 \to R_3$$

$-R_1+R_3 \to R_3$

$$= 2\begin{vmatrix} a & b & c \\ 2d+g & 2e+h & 2f+i \\ 2g & 2h & 2i \end{vmatrix} + \begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} \; R_1 \leftrightarrow R_3, \; R_2 \leftrightarrow R_3$$

$$= 4\begin{vmatrix} a & b & c \\ 2d+g & 2e+h & 2f+i \\ g & h & i \end{vmatrix} + \underbrace{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}_{= 1}$$

$$= 8\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + 1 = 8 + 1 = 9$$

Before Class Discussion

(1 point) **Library/Hope/Multi1/04-02-Kernel-image/Ker_im_12.pg**

Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$f(x,y,z) = \begin{bmatrix} -5 \\ -12 \end{bmatrix} x + \begin{bmatrix} -2 \\ -5 \end{bmatrix} y + \begin{bmatrix} 12 \\ 29 \end{bmatrix} z.$$

Find bases for the kernel and image of f. vector

A basis for the kernel of f is $\{ \quad \}$.

A basis for the image of f is $\{ \quad \}$.

$\mathrm{Ker}(f) = \left\{ (x,y,z) \middle| \begin{pmatrix} -5x - 2y + 12z \\ -12x - 5y + 29z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

A

$$\begin{pmatrix} -5 & -2 & 12 \\ -12 & -5 & 29 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/5 & 12/5 \\ -12 & -5 & 29 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/5 & 12/5 \\ 0 & -1/5 & \frac{289}{5} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2/5 & 12/5 \\ 0 & 1 & -289 \end{pmatrix}$$

$x + \frac{2}{5}y + \frac{12}{5}z = 0$ $x = -\frac{2}{5}y - \frac{12}{5}z$

$y - 289z = 0$ $y = 289z$

$x = \left(-\frac{2}{5}(289) - \frac{12}{5} \right) z$

$\mathrm{Ker}(f) = \{ (\downarrow , 289z, z) \mid z \in \mathbb{R} \}$ $= z\boxed{\left(-\frac{2}{5}(289) - \frac{12}{5}, 289, 1 \right)}$

Rank Nullity: $\dim(\mathbb{R}^3) = \dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f)$

$3 = 1 + 2$

but $\mathrm{Im}(f) \subseteq \mathbb{R}^2$ since they have the same dimension. $\{(1,0), (0,1)\}$ is a basis for $\mathrm{Im}(f)$

$f(x,y,z) = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so $\mathrm{Im}(f) = \mathrm{colspace}(A)$ so we can find a basis for the column space by using "standard procedure"

$$A \sim \begin{pmatrix} 1 & 2/5 & 12/5 \\ 0 & 1 & -289 \end{pmatrix}$$

since there are leading 1s in the 1st and 2nd columns,

a basis for colspace(A) = Im(f) is the first two columns of A

namely $\left\{ \begin{pmatrix} -5 \\ -12 \end{pmatrix}, \begin{pmatrix} -2 \\ -5 \end{pmatrix} \right\}$

Beginning of the lecture.

$$A = \begin{pmatrix} -3 & 2 & 3 \\ -4 & 3 & 6 \\ 3 & -3 & 1 \end{pmatrix}$$

Find all eigenvalues and eigenspace.
Is A defective or non-defective?

$$\det(A - \lambda I) = \left| \begin{array}{ccc|cc} -3-\lambda & 2 & 3 & -3-\lambda & 2 \\ -4 & 3-\lambda & 6 & -4 & 3-\lambda \\ 3 & -3 & 1-\lambda & 3 & -3 \end{array} \right.$$

$$= (-3-\lambda)(3-\lambda)(1-\lambda) + 36 + 36 - [9(3-\lambda) + 18(+3+\lambda) - 8(1-\lambda)]$$

$$= -(\lambda+3)(\lambda-3)(\lambda-1) + 72 - (27 - 9\lambda + 54 + 18\lambda - 8 + 8\lambda)$$

$$= -(\lambda+3)(\lambda-3)(\lambda-1) + 72 - (73 + 17\lambda)$$

$$= -(\lambda+3)(\lambda-3)(\lambda-1) - 1 - 17\lambda = -(\lambda^2-9)(\lambda-1) - 1 - 17\lambda$$

$$= -(\lambda^3 - \lambda^2 - 9\lambda + 9) - 1 - 17\lambda = -\lambda^3 + \lambda^2 - 8\lambda - 10 = p(\lambda)$$

We want to set $p(\lambda) = 0$ and factorize $p(\lambda)$.

$p(-1) = 1 + 1 + 8 - 10 = 0$ therefore $\lambda - (-1) = \lambda + 1$ is a factor of $p(\lambda)$

So $-\lambda^3 + \lambda^2 - 8\lambda - 10 = -(\lambda+1)(\underset{\substack{\| \\ 1}}{a}\lambda^2 + b\lambda + \underset{\substack{\| \\ 10}}{c}) = -(\lambda+1)(\lambda^2 + b\lambda + 10)$

To figure out the value of b, you can set the coefficient of λ^2 on both sides equal to each other. Or in this case plugging in $\lambda = 1$ is easier.

$\lambda=1$ $\qquad\qquad\qquad\qquad\qquad\qquad \lambda=1$

$$-1 + 1 - 8 - 10 = -18 = -(2)(1 + b + 10) = -2(11 + b)$$

So $11 + b = 9$ $\qquad b = -2$

So now we got $p(\lambda) = -(\lambda+1)(\lambda^2 - 2\lambda + 10) = 0$

$\lambda + 1 = 0$ or $\lambda = -1$, or $\lambda^2 - 2\lambda + 10 = 0$

$$= (\lambda-1)^2 - 1 + 10 = (\lambda-1)^2 + 9 = 0$$

$$(\lambda-1)^2 = -9$$

$$\lambda - 1 = \pm 3i \qquad \lambda = 1 \pm 3i$$

Eigenvalues, $\lambda = -1$, $1 \pm 3i$

Next we are looking for corr. eigenvectors.

$\lambda = -1$

$$A - \lambda I = \begin{pmatrix} -2 & 2 & 3 \\ -4 & 4 & 6 \\ 3 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -3/2 \\ 3 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & -3/2 \\ 0 & 0 & 2-3(-3/2) \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & -1 & -3/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} x_1 & x_2 & x_3 \\ \text{bound} & \text{free} & \text{bound} \end{array} \end{array}$$

$x_1 - x_2 - \cancel{3/2\, x_3} = 0$

$x_3 = 0$

$x_1 = x_2 = t$

$$E_{-1} = \{(t, t, 0) \mid t \in \mathbb{C}\} \qquad t(1, 1, 0)$$

$\lambda = 1 + 3i$

$$A - \lambda I = \begin{pmatrix} -4-3i & 2 & 3 \\ -4 & 2-3i & 6 \\ 3 & -3 & -3i \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -i \\ -4-3i & 2 & 3 \\ -4 & 2-3i & 6 \end{pmatrix}$$

$(4+3i)R_1 + R_2 \to R_2$

$4R_1 + R_3 \to R_3$

$$\sim \begin{pmatrix} 1 & -1 & -i \\ 0 & -2-3i & 6-4i \\ 0 & -2-3i & 6-4i \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -i \\ 0 & 1 & -2i \\ 0 & 0 & 0 \end{pmatrix}$$

$\frac{1}{-2-3i} R_2$

$$\frac{1}{-2-3i} \cdot (6-4i) = \frac{(-2+3i)}{(-2+3i)} \frac{(6-4i)}{(-2-3i)} = \frac{-26i}{13} = -2i$$

$x_3 = t$

$x_1 - x_2 - ix_3 = 0$

$x_2 - 2ix_3 = 0 \to x_2 = 2it \qquad x_1 = x_2 + ix_3 = 2it + it = 3it$

$$E_{1+3i} = \{(3it, 2it, t) \mid t \in \mathbb{C}\} \qquad t(3i, 2i, 1)$$

so

$$E_{\overline{1+3i}} = E_{1-3i} = \{\overline{(3i, 2i, 1)}\, t \mid t \in \mathbb{C}\} = \{(-3i, -2i, 1)\, t \mid t \in \mathbb{C}\}$$

A is non-defective since the sum of the dimensions of eigenspaces add up to $\dim(\mathbb{C}^3) = 3$.

Example Say $Av = -2v$ and A is invertible. $(v \neq 0)$
Find $A^3 v$, $A^{-1} v$, $(A + 5I_n)v$, $10Av$ and $(A^4 + 5A^2 - 10A - 2I + A^{-1})v$.

$$A^3 v = A^2(Av) = A^2(-2v) = -2A^2 v = -2A(Av)$$

$$= 4Av = -8v \qquad (\text{so } A^3 v = (-2)^3 v)$$

So v is an eigenvector of A^3 with eigenvalue $(-2)^3$.

$A^{-1} v$ $\qquad \underbrace{A^{-1}(Av)}_{v} = A^{-1}(-2v) = -2A^{-1}v$

$$A^{-1} v = \frac{-1}{2} v = (-2)^{-1} v$$

$(A+5I_n)v = Av + 5I_n v = -2v + 5v = 3v = (-2+5)v$

$10Av = 10(-2v) = -20v = \underline{\underline{(-2)\cdot 10}}\,v$

$(A^4 + 5A^2 - 10A - 2I + A^{-1})v = (-2)^4 v + 5(-2)^2 v - 10(-2)v - 2v + (-2)^{-1}v$

$= \left((-2)^4 + 5(-2)^2 - 10(-2) - 2 + (-2)^{-1}\right)v$

$= \left(16 + 20 + 20 - 2 - \frac{1}{2}\right)v = \left(53 + \frac{1}{2}\right)v$

Higher order linear diff. operator

$y' = Dy \qquad D^2 y = D(Dy) = Dy' = y''$

$y''' - 5y'' + 10y = D^3 y - 5D^2 y + 10y$

$= \underbrace{(D^3 - 5D^2 + 10)}_{L}\,y = Ly$ ← define L this way.

If L is a linear diff. operator of degree n then

$Ker(L) = \{ Ly = 0 \mid y$ is at least n-times differentiable and $y^{(n)}$ is cont.$\}$

is n-dimensional $\dim(Ker L) = n$.

Example Find all the solutions to $y'' - 2y' - 15y = 0$ Hint use

$y = e^{rx} \qquad y' = re^{rx} \qquad y'' = r^2 e^{rx}$

So $y'' - 2y' - 15y = r^2 e^{rx} - 2re^{rx} - 15e^{rx} = 0$

$e^{rx}(r^2 - 2r - 15) = 0$

so $r^2 - 2r - 15 = 0$

$(r-5)(r+3) = 0$ so $r = -3, 5$

So actually $\underline{e^{-3x}}$ and $\underline{e^{5x}}$ are solutions to the diff. eq.

let $L = D^2 - 2D - 15$ then $y'' - 2y' - 15y = 0$ is $Ly = 0$

Since L is degree 2, $\dim(Ker L) = 2$. Since $e^{-3x}, e^{5x} \in Ker L$

and they are LI, they form a basis for $Ker L$.

$Ker L = \{ ae^{-3x} + be^{5x} \mid a, b \in \mathbb{R} \}$

After Class Discussion

$A = \begin{bmatrix} 4 & -6 \\ 6 & -9 \end{bmatrix}_{2\times 2}.$

Find bases for the kernel and image of $T(\vec{x}) = A\vec{x}$.

A basis for the kernel of A is $\left\{ \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} \right\}$.

A basis for the image of A is $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Rank-Nullity $\dim(\mathbb{R}^2) = \dim(Ker T) + \dim(Im T)$

$0 + 2$

$2 = 1 + 1$

$Ker T \subseteq \mathbb{R}^2$

$2 + 0$

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

As long as $\det A \neq 0$, $\Rightarrow$ $nullspace(A) = \{ x \mid Ax = 0 \} = \{0\}$

$nullspace(A) = Ker(T)$

$A^{-1}Ax = A^{-1}0$

$x = 0$

So $\dim(Ker T) = 0$ only when $\det(A) \neq 0$ Remark of course this only works for square matrices.

(1 point) Let $V = \mathbb{R}^{2\times 2}$ be the vector space of 2×2 matrices and let $\underline{L : V \to V}$ be defined by $L(X) = \begin{bmatrix} -6 & -2 \\ 9 & 3 \end{bmatrix} X$. Hint: The image of a spanning set is a spanning set for the image.

a. Find $L\left(\begin{bmatrix} -3 & -2 \\ 1 & -2 \end{bmatrix}\right) = \begin{bmatrix} 16 & 16 \\ -24 & -24 \end{bmatrix}$

b. Find a basis for $\ker(L)$:

$\left\{ \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \right\}$

c. Find a basis for $\operatorname{ran}(L)$:

$\left\{ \begin{bmatrix} -6 & 0 \\ 9 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix} \right\}$

$\dim(V) = \dim(M_2\mathbb{R}) = 4$

$\dim(V) = \dim(Ker L) + \dim(Rng L)$

$4 = \begin{cases} 0 + 4 \\ 1 + 3 \\ 2 + 2 \\ 3 + 1 \\ 4 + 0 \end{cases}$

$L(X) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} X \qquad L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$

$T: \mathbb{R}^4 \to \mathbb{R}^4$

$T(a, b, c, d) = (a, b, a, b)$

$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

$T(a,b,c,d) = A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$M_2(\mathbb{R}) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \overset{\mathbb{R}^4}{(a, b, c, d)} \longleftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mathbb{R}^4$

$L: M_2(\mathbb{R}) \to M_2(\mathbb{R}) \Longleftrightarrow T: \mathbb{R}^4 \to \mathbb{R}^4$

#24 $\quad A_{2\times 2} \qquad T(\overset{\text{vector}}{\underset{2\times 1}{x}}) = Ax_{2\times 1} \qquad \underset{\mathbb{R}^4}{T(x)} = \underbrace{A_{5\times 4}\,x_{4\times 1}}_{5\times 1}$

#23 $\quad \mathbb{R}^{2\times 2} = M_2(\mathbb{R}) \qquad L(X) = \begin{pmatrix} -6 & -2 \\ 9 & 3 \end{pmatrix} X_{2\times 2} \qquad T(x) \in \mathbb{R}^5$

$L: V \to V$