I will go over some of the difficult problems from 1pm exam on my Zoom office hours (Thursday 1-2pm). They will be recorded.

Eigenvalues / Eigenvectors

$A \in M_n(\mathbb{R})$ and $x \in \mathbb{R}^n$ $(x \neq 0)$ if there is a scalar λ such that $Ax = \lambda x$ then x is called an eigenvector of A corresponding to the eigenvalue λ.

Note: If $Ax = \lambda x$ for $x \neq 0$ then for any nonzero scalar k, we have $A(kx) = \lambda(kx)$. So kx is another eigenvector corr. to the eigenvalue λ.

Example $A = \begin{pmatrix} 5 & 2 \\ 0 & 7 \end{pmatrix}$ Set $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 5 & 2 \\ 0 & 7 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 5-\lambda & 2 \\ 0 & 7-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 0 & 7-\lambda \end{vmatrix} = (5-\lambda)(7-\lambda) = (\lambda-5)(\lambda-7) = 0$$

$\lambda = 5, 7$

To find the eigenvectors corresponding to λ, we solve the eq $(A - \lambda I)x = 0$. For example,

for $\lambda = 5$, $A - 5I = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$ $0x_1 + 2x_2 = 0 \Downarrow x_2 = 0$

So x_1 is a free variable and $x_2 = 0$

So eigenvectors are of the form $\begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $t \neq 0$

$\lambda = 7$

$A - 7I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ $x_1 - x_2 = 0$, x_2 is a free var

and $x_1 = x_2 = t$ so eig. vecs look like $\begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $t \neq 0$

Example $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda^2 = -1$ $\lambda = \pm i$ $(A - \lambda I)x = 0$

$\lambda = i$ $A - \lambda I = A - iI = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \sim \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \xrightarrow{iR_1 + R_2 \to R_2} \begin{pmatrix} 1 & i \\ -i+i & 1-1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ $x_1 + ix_2 = 0$ $(x_2 = t$ free$)$

So $x_1 = -ix_2 = -it$

So $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -it \\ t \end{pmatrix} = t\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector (for $t \neq 0$) corr. to the eig. value $\lambda = i$.

Since we have a pair of complex conjugate eigenvalues $(\bar{i} = -i)$, the eigenvectors for $\bar{i} = -i$ are of the form

$$t\overline{\begin{pmatrix} -i \\ 1 \end{pmatrix}} = t\begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{for } t \neq 0.$$

Example $A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} = (4-\lambda)^2$

$= (\lambda - 4)^2$ so the eigenvalue $\lambda = 4$ is repeated twice.

$\lambda = 4$

$A - \lambda I = A - 4I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $x_2 = 0$ and x_1 is a free variable

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $(t \neq 0)$ are all the eigenvectors

Eigenspace for $\lambda = 4$ is $E_4 = \{ x \in \mathbb{R}^2 \mid Ax = 4x \}$

(so all the eigenvectors for $\lambda = 4$ and the zero vector)

$\dim(E_4) = 1$

Defective and Non-Defective Matrices

An $n \times n$ matrix A is called non-defective if there is a linearly independent set of eigenvectors for A which is a basis for $\mathbb{C}^n$ (or if the set has exactly n elements)

So from the examples above we see that

$A = \begin{pmatrix} 5 & 2 \\ 0 & 7 \end{pmatrix}$ is non-defective since $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{C}^2$ (using complex scalars).

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is non-defective since $\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{C}^2$.

$A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$ is defective because any two eigenvectors are linearly dependent.

Example $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

upper triangular so det is the product of the terms on the diagonal.

$= (1-\lambda)(2-\lambda)(3-\lambda) = -(\lambda-1)(\lambda-2)(\lambda-3)$

So the eig. values are $1, 2, 3$.

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_2 = 0$
$x_3 = 0$
x_1 is free.

eig. vectors look like $\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_2 = 0$
$x_3 = 0$
x_2 is free
and $x_1 = x_2 = t$

so $X = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 3$

$$A - \lambda I = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - \frac{1}{2} x_2 = 0$
$x_2 + x_3 = 0$
x_3 is free.
$x_3 = t$

$x_2 = -x_3 = -t$

$x_1 = \frac{1}{2} x_2 = -\frac{t}{2}$

$$X = \begin{pmatrix} -t/2 \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}$$

A is automatically non-defective because eigenvectors that correspond to distinct eigenvalues are automatically linearly independent and we have 3 distinct eig. values. So

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix} \right\}$ are 3 LI vectors in a 3-dimensional space. So they form a basis. So A is non-defective.

Example

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 0 & 3-\lambda & 0 \\ 1 & -1 & 2-\lambda \end{vmatrix}$$

(Cofactor expansion along 2^{nd} row)

$$= (3-\lambda) \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)^2 = -(\lambda-3)(\lambda-2)^2$$

$\lambda = 3$

$$A - \lambda I = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - x_2 - x_3 = 0$
$x_2 - \frac{1}{2} x_3 = 0 \rightarrow x_2 = \frac{1}{2} x_3 = \frac{1}{2} t$
x_3 is free $(x_3 = t)$
$x_1 = x_2 + x_3 = \frac{1}{2} t + t = \frac{3}{2} t$

$$X = \begin{pmatrix} \frac{3}{2} t \\ \frac{1}{2} t \\ t \end{pmatrix} = t \begin{pmatrix} 3/2 \\ 1/2 \\ 1 \end{pmatrix} \qquad E_3 = \left\{ t \begin{pmatrix} 3/2 \\ 1/2 \\ 1 \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = 0$
$x_2 = 0$
x_3 is free

$$X = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{so } E_2 = \left\{ t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$$

Since $\dim(E_3) + \dim(E_2) = 1 + 1 = 2 < 3 = \dim(\mathbb{C}^3)$,

A is defective.