Discussion before class

$e^{it} = \cos t + i \sin t$

$e^{-it} = \cos t - i \sin t$

$e^{it}, e^{-it} \in \text{span}\{\cos t, \sin t\}$

$\frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos t$

$\cos t, \sin t \in \text{span}\{e^{it}, e^{-it}\}$

$\frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it} = \sin t$

The beginning of the lecture

Example Given linear $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (2x_1 - x_2, 5x_2, x_1 + 3x_2)$ find the matrix A s.t. $T(x) = Ax$

$T(1,0) = (2, 0, 1) \qquad T(0,1) = (-1, 5, 3)$

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 5 \\ 1 & 3 \end{pmatrix}.$$

Kernel and Range of $T: V \to W$

$\text{Ker}(T) = \{ v \in V \mid T(v) = 0 \}$ subspace of V

$\text{Rng}(T) = \{ T(v) \mid v \in V \}$ subspace of W

For $T: \mathbb{R}^n \to \mathbb{R}^m \longleftrightarrow A_{m \times n}$

$\text{Ker}(T) \longleftrightarrow \text{nullspace}(A)$

$\text{Rng}(T) \longleftrightarrow \text{colspace}(A)$

However, Kernel and Range are defined for any linear transformation (not just $T: \mathbb{R}^n \to \mathbb{R}^m$)

Generalized Rank Nullity: for $T: V \to W$

$\dim(\text{Ker}(T)) + \dim(\text{Rng}(T)) = \dim V$

Example Determine the kernel and the range of $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (2x_1 - x_2, 5x_2, x_1 + 3x_2)$

$\text{Ker}(T) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid T(x_1, x_2) = 0 \}$

$(2x_1 - x_2, 5x_2, x_1 + 3x_2) = (0,0,0)$

$2x_1 - x_2 = 0$

$5x_2 = 0 \longrightarrow x_2 = 0$

$x_1 + 3x_2 = 0 \longrightarrow x_1 = 0$

$\text{Ker}(T) = \{(0,0)\}$ (so it is 0-dimensional)

$\text{Rng}(T) = \{ T(x_1, x_2) \mid x_1, x_2 \in \mathbb{R} \}$

$= \{ (2x_1 - x_2, 5x_2, x_1 + 3x_2) \mid x_1, x_2 \in \mathbb{R} \}$

$(2x_1 - x_2, 5x_2, x_1 + 3x_2) = x_1(2, 0, 1) + x_2(-1, 5, 3)$

$\{(2,0,1), (-1,5,3)\}$ is a spanning set for $\text{Rng}(T)$

Further, they are LI so they form a basis for $\text{Rng}(T)$.

So $\dim(\text{Rng}(T)) = 2$. $\quad$ ($\underset{0}{\dim(\text{Ker}\,T)} + \underset{2}{\dim(\text{Rng}\,T)} = \underset{2}{\dim V = \dim \mathbb{R}^2}$) ✓

Example Find bases for $\text{Ker}(T)$ and $\text{Rng}(T)$ where $T: P_2(\mathbb{R}) \to \mathbb{R}^2$ is defined by $T(p) = (p(1), p''(5) - p'(0))$

$\text{Ker}(T) = \{ p \in P_2(\mathbb{R}) \mid T(p) = 0 \}$

$p(x) = a + bx + cx^2 \qquad p'(x) = b + 2cx \qquad p''(x) = 2c$

$T(p) = (p(1), p''(5) - p'(0)) = (a+b+c, 2c - b) = 0$

$\text{Ker}(T) = \{ a + bx + cx^2 \mid a+b+c = 0 \quad 2c - b = 0 \}$

$b = 2c$

$a = -b - c = -2c - c = -3c$

$\text{Ker}(T) = \{ \underset{c(-3+2x+x^2)}{-3c + 2cx + cx^2} \mid c \in \mathbb{R} \}$ so $\{-3 + 2x + x^2\}$ is a basis for $\text{Ker}(T)$.

So $\dim(\text{Ker}(T)) = 1$. We also know $\dim(P_2 \mathbb{R}) = 3$

So by the generalized rank nullity $\dim(\text{Rng}\,T) = 3 - 1 = 2$

(Note that $\text{Rng}(T)$ is a subspace of $\mathbb{R}^2$ of dimension 2. So in fact $\text{Rng}(T) = \mathbb{R}^2$ so $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$)

Since $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$, $\{T(1), T(x), T(x^2)\}$ is a spanning set for $\text{Rng}(T) \subseteq \mathbb{R}^2$.

$T(p) = (p(1), p''(5) - p'(0))$

$T(1) = (1, 0) = w_1$

$T(x) = (1, -1) = w_2$

$T(x^2) = (1, 2) = w_3$

$c_1 w_1 + c_2 w_2 + c_3 w_3 = (0,0)$

$-3w_1 + 2w_2 + w_3 = (-3,0) + (2,-2) + (1,2) = (0,0)$

So $w_3 = 3w_1 - 2w_2$ and we can get rid of w_3 without changing the span.

$\{w_1, w_2\}$ is a spanning set for $\text{Rng}(T)$.

Since $\dim(\text{Rng}\,T) = 2$, this set must be a basis.

Example Find bases for $Ker(T)$ and for $Rng(T)$ where $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ $\quad T(A) = A + A^T$

$Ker(T) = \{ A \in M_2(\mathbb{R}) \mid T(A) = A + A^T = 0 \} = \left\{ \begin{matrix} \text{skew-symmetric} \\ \text{matrices} \end{matrix} \right\}$

$A^T = -A$

$$Ker(T) = \left\{ \underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \;\middle|\; \underset{A^T}{\begin{pmatrix} a & c \\ b & d \end{pmatrix}} = \underset{-A}{-\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \right\}$$

$a = -a \longrightarrow 2a = 0 \quad a = 0$

$b = -c$

$d = -d \longrightarrow 2d = 0 \quad d = 0$

$c = -b$

$$= \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\} \quad \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ is a basis for $Ker(T)$. (So $\dim(Ker\, T) = 1$.)

$\dim(M_2 \mathbb{R}) = 4$ so $\dim(Rng(T)) = 4 - 1 = 3$ (Rank-Nullity)

$$Rng\,T = \{ T(A) = A + A^T \mid A \in M_2(\mathbb{R}) \} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} = a\underline{\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}} + b\underline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + c\underline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + d\underline{\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}}$$

So $\left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\}$ is a basis for $Rng(T)$.

After class discussion

$x^2 + 2x + 17 = (x+1)^2 + 16 = 0$

$(x+1)^2 = -16$

$x + 1 = \pm\sqrt{-16} = \pm 4i$

$x = -1 \pm 4i$

$x^3 - 6x^2 + 11x - 6 \qquad 1 - 6 + 11 - 6 = 0$ so $(x-1)$ is a factor of the cubic.

$\cancel{x = 0} \qquad x = 1$

$x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 + kx + 6)$

$x^3 \qquad -6$

$-6x^2 = -x^2 + kx^2 = (k-1)x^2$

$-6 = k - 1$

$-5 = k$

$x^3 - 6x^2 - 11x - 6 = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$

-2
-3