Def^n $T: V \to W$ is a linear transformation

if $T(u+v) = T(u) + T(v)$ for all vectors u and v in V

and $T(cv) = cT(v)$ for all vectors v in V and scalars c.

Fact $\underset{\text{in } V}{T(0)} = \underset{\text{in } W}{0}$

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k)$$

Questions / Example

1) Is $T: M_n(\mathbb{R}) \to \mathbb{R}$, $T(A) = \det(A)$ a linear transformation?

for $n=1$ $A = (a)$ $T(A) = \det(A) = \det(a) = a$

$A = (a)$, $B = (b)$ $T(A+B) = \det(a+b) = a+b$
$= T(A) + T(B)$

$T(cA) = \det(ca)$
$= c\det(a) = cT(A)$ ✓

for $n > 1$ $T(A) = \det(A)$ is not linear!

$T(cA) = \det(cA) = c^n\det(A) = c^n T(A)$

$c = 2$ $A = I_n$

$T(2I) = \det(2I) = 2^n\det(I) = 2^n T(I) \neq 2T(I)$ $n>1$

2) $T: C^0[a,b] \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x)dx$

($C^0[a,b]$: continuous function on $[a,b]$)

Is T linear?

$T(f+g) = \int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx = T(f) + T(g)$

$T(cf) = \int_a^b cf(x)dx = c\int_a^b f(x)dx = cT(f)$ ✓

3) $T: M_2(\mathbb{R}) \to P_2(\mathbb{R})$ Is T linear?

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{2a-b}_{\text{const}} + \underbrace{(c+d)x}_{(\text{const})\,x} \underbrace{- ax^2}_{(\text{const})x^2}$$

$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \overset{?}{=} T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we want to show that

$= T\begin{pmatrix} a+A & b+B \\ c+C & d+D \end{pmatrix} = 2(a+A) - (b+B) + (c+C+d+D)x - (a+A)x^2$

$= 2a - b + (c+d)x - ax^2 + 2A - B + (C+D)x - Ax^2$

$= T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ ✓

$T\left(k\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = T\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \overset{?}{=} kT\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= 2(ka) - kb + (kc+kd)x - kax^2$

$= k\left[2a - b + (c+d)x - ax^2\right] = kT\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ✓

Therefore, T is linear.

4) $T: P_2(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$

$T(p) = \begin{pmatrix} p(0) & p'(1) & 0 \\ 2p(1) - p''(2) & 0 & 0 \end{pmatrix}$ is this linear?

$T(p+q) = \begin{pmatrix} (p+q)(0) & (p+q)'(1) & 0 \\ 2(p+q)(1) - (p+q)''(2) & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} p(0)+q(0) & p'(1)+q'(1) & 0 \\ 2p(1) + 2q(1) - p''(2) - q''(2) & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} p(0) & p'(1) & 0 \\ 2p(1) - p''(2) & 0 & 0 \end{pmatrix} + \begin{pmatrix} q(0) & q'(1) & 0 \\ 2q(1) - q''(2) & 0 & 0 \end{pmatrix}$

$= T(p) + T(q)$ ✓

$T(cp) = \begin{pmatrix} (cp)(0) & (cp)'(1) & 0 \\ 2(cp)(1) - (cp)''(2) & 0 & 0 \end{pmatrix} = c\begin{pmatrix} p(0) & p'(1) & 0 \\ 2p(1) - p''(2) & 0 & 0 \end{pmatrix}$

$= cT(p)$ ✓

T is linear.

4') what about $L(p) = \begin{pmatrix} p(0) & p'(1) \\ 0 & 1 \end{pmatrix}$?

$L(\underset{\text{the 0 polynomial}}{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$ in $M_2(\mathbb{R})$

so it is not a linear map.

5) Determine the linear transformation $T: P_2(\mathbb{R}) \to \mathbb{R}^4$ defined by $T(1) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $T(x) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $T(x^2) = \begin{pmatrix} -5 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

$\{1, x, x^2\}$ form a basis for $P_2(\mathbb{R})$.

$$T(a1 + bx + cx^2) = aT(1) + bT(x) + cT(x^2) = a\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} -5 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2a - 5c \\ a + b \\ b - c \\ b \end{pmatrix}$$

6) Determine the $\underline{\text{matrix transformation}}$ defined by the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ -1 & -1 \end{pmatrix}_{3\times 2}$ $\Big(T(x) = Ax \Big)$

$x_{2\times 1} \in \mathbb{R}^2$

Then $T(x) = (Ax)_{3\times 1} \in \mathbb{R}^3$

So $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ 5y \\ -x - y \end{pmatrix}$$

$$T(x,y) = (2x+y,\ 5y,\ -x-y)$$

7) Given that linear $T: \mathbb{R}^2 \to P_3(\mathbb{R})$ satisfies

$T(2,1) = 1 - x^3$ and $T(5,-1) = 2 + x^2$

Find $T(1,4)$. Note that $\{(2,1), (5,-1)\}$ form a basis for $\mathbb{R}^2$.

So we can find c_1, c_2 such that $\boxed{(1,4) = c_1(2,1) + c_2(5,-1)}$

In that case,

$$T(1,4) = T(c_1(2,1) + c_2(5,-1)) = c_1 T(2,1) + c_2 T(5,-1)$$

$$= c_1(1 - x^3) + c_2(2 + x^2) = \underline{c_1 + 2c_2 + c_2x^2 - c_1x^3}$$

So it is enough to figure out c_1 and c_2.

$(1,4) = c_1(2,1) + c_2(5,-1) = (2c_1 + 5c_2,\ c_1 - c_2)$

So $\quad 2c_1 + 5c_2 = 1$

$\quad\quad c_1 - c_2 = 4 \quad \rightsquigarrow A^{\#} = \left(\begin{array}{cc|c} 2 & 5 & 1 \\ 1 & -1 & 4 \end{array}\right)$

$$A^{\#} \sim \left(\begin{array}{cc|c} 1 & -1 & 4 \\ 2 & 5 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 7 & -7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & -1 \end{array}\right)$$

$c_1 - c_2 = 4$

$c_2 = -1$ $\longrightarrow$ so $c_1 = 4 + c_2 = 4 - 1 = 3$

$$\underline{T(1,4)} = c_1 + 2c_2 + c_2x^2 - c_1x^3 = \underline{1 - x^2 - 3x^3}$$