DISCUSSION BEFORE CLASS

$f_1 = \sin x \quad f_2 = \cos x \quad f_3 = \tan x \qquad \cos^2 + \sin^2 = 1$

$$W(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ -\sin x & -\cos x & 2\sec^2 x \tan x \end{vmatrix} \qquad W(0) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = 0$$

$$W\left(\frac{\pi}{4}\right) = \left(\frac{1}{2}\right)^3 \begin{vmatrix} \sqrt{2} & \sqrt{2} & 2 \\ \sqrt{2} & -\sqrt{2} & 1 \\ -\sqrt{2} & -\sqrt{2} & 2 \end{vmatrix} = \left(\frac{1}{2}\right)^3 \begin{vmatrix} \sqrt{2} & \sqrt{2} & 2 \\ \sqrt{2} & -\sqrt{2} & 1 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)^3 \left[4 \begin{vmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{vmatrix} \right] \neq 0 \Rightarrow f_1, f_2, f_3 \text{ are LI}$$

$W(x) = | \ldots | = x^2 \neq 0 \qquad W(1) = 1 \neq 0$

The Beginning of the Lecture

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \qquad c_1 = c_2 = \cdots = c_n = 0$

If this is the only solution, $v_1, \ldots, v_n$ are LI. Otherwise LD.

$$\begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{pmatrix}_{m \times n} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{n \times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m \times 1}$$

Example Find a basis for $W = \text{span}\{(2,-1), (-4,2), (3,5)\}$

Let $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \\ 3 & 5 \end{pmatrix}$. By definition rowspace $(A) = W$.

$$A \sim \begin{pmatrix} 2 & -1 \\ 0 & 0 \\ 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 3 & 5 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 13/2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\{(1, -\frac{1}{2}), (0,1)\}$ is a basis for rowspace $(A) = W$.

Note that these vectors do not form a subset of the vectors that we started with.

Say we want a basis which is a subset of $\{(2,-1), (-4,2), (3,5)\}$

Let $A = \begin{pmatrix} 2 & -4 & 3 \\ -1 & 2 & 5 \end{pmatrix}$ (so that colspace$(A) = W$)

$$A \sim \begin{pmatrix} 1 & -2 & -5 \\ 2 & -4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -5 \\ 0 & 0 & 13 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

leading ones in 1^{st} and 3^{rd} columns

so the 1^{st} and 3^{rd} columns of A give us a basis for colspace $(A) = W$.

$\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$ or $\{(2,-1), (3,5)\}$ is a basis for W which is a subset of the initial vectors that we started with.

Example Find a basis for $W = \text{span}\{2 - x^2, -4 + 2x^2, 3 + 5x^2\}$

$P_2(\mathbb{R}) \ni a + bx + cx^2 \iff (a, b, c) \in \mathbb{R}^3$

$2 - x^2 = 2 + 0x + (-1)x^2 \iff (2, 0, -1)$

$-4 + 2x^2 = -4 + 0x + 2x^2 \iff (-4, 0, 2)$

$3 + 5x^2 = 3 + 0x + 5x^2 \iff (3, 0, 5)$

$$A = \begin{pmatrix} 2 & 0 & -1 \\ -4 & 0 & 2 \\ 3 & 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} 1 + 0x - \frac{1}{2}x^2 = 1 - \frac{x^2}{2} \\ 0 + 0x + 1x^2 = x^2 \end{matrix}$$

So $\{1 - \frac{x^2}{2}, x^2\}$ is a basis for W.

Rank-Nullity Theorem

$A_{m \times n}$ then rank(A) + nullity$(A) = n$

$A_{m \times n} X_{n \times 1} = 0$

rank(A) = # leading ones = # bound variables

nullity(A) = dim(nullspace(A)) = # free variables

n = # columns = # all the variables

Example $A = \begin{pmatrix} 1.5 & -6 & 9 \\ -1 & 4 & -6 \end{pmatrix}_{2 \times 3} \sim \begin{pmatrix} 1 & -4 & 6 \\ 1.5 & -6 & 9 \end{pmatrix}$

$-\frac{3}{2}R_1 + R_2 \to R_2$

$$\sim \begin{pmatrix} 1 & -4 & 6 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow x_1 - 4x_2 + 6x_3 = 0$$

the only leading one; rank$(A) = 1$

x_2 and x_3 are free variables. $x_2 = s \quad x_3 = t$

then $x_1 = 4s - 6t$

So nullspace$(A) = \{x \in \mathbb{R}^3 \mid Ax = 0\}$

$= \{(x_1, x_2, x_3) = (4s - 6t, s, t) \mid s, t \in \mathbb{R}\}$

$= \{(4s - 6t, s, t) \mid s, t \in \mathbb{R}\}$

$(4s - 6t, s, t) = s(4, 1, 0) + t(-6, 0, 1)$ so

$\{(4, 1, 0), (-6, 0, 1)\}$ is a spanning set for nullspace(A).

In fact it is LI so it is a basis for nullspace (A)

and thus, $\text{nullity}(A) = \dim(\text{nullspace}(A)) = 2$.

$\text{rank}(A) + \text{nullity}(A) = 1 + 2 = 3 = n$

$\text{nullity}(A) = n - \text{rank}(A) = 3 - 1 = 2$

Example let $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} -6 \\ 0 \\ -2 \end{pmatrix}$

Verify that $x_p = (-1, 1, -1)$ is a particular solution to $Ax = b$ and find the general solution to $Ax = b$.

$x_{general} = x_{particular} + x_{homogeneous}$ $(Ax = 0)$

$Ax_p = b$? $Ax_p = \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 + 1 - 5 \\ -1 - 1 + 2 \\ 0 + 1 - 3 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -2 \end{pmatrix} = b$ ✓

Say x_p and y are two solutions to $Ax_p = b$, $Ay = b$

$b - b = Ay - Ax_p = A(y - x_p) = 0$ so $y - x_p \in \text{nullspace}(A)$

$b - b = 0$

$y - x_p = x_{homogeneous}$

$y_{general} = x_{particular} + x_{homogeneous}$

(↑ general solution to the homogeneous part)

So we just need to solve $Ax = 0$

$A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 2 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 3 & 9 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 - x_2 - 2x_3 = 0$

$x_2 + 3x_3 = 0$

bound variables; x_3 is free

set $x_3 = t \in \mathbb{R}$

$x_2 = -3t$ $x_1 = x_2 + 2x_3 = -3t + 2t = -t$

$\text{nullspace}(A) = \{ (-t, -3t, t) \mid t \in \mathbb{R} \}$

$(-t, -3t, t) = t(-1, -3, 1)$

so the general solution to $Ax = b$ is

$x_{general} = x_p + x_h = (-1, 1, -1) + t(-1, -3, 1)$

AFTER CLASS DISCUSSION

A is inv. $\iff$ $Ax = b$ has a uniq. solution

$\Downarrow$

$\text{nullspace}(A) = \{(0,0,0)\}$

$x_{general} = x_p + x_h = x_p + 0 = x_p$