Example Say $\{v_1, v_2, v_3\}$ is a spanning set for a v.s. V and $\dim(V) = 3$. Show that $\{v_1, v_2, v_3\}$ is LI
(Of course, once it is LI, it is a basis.)

Pf Assume that $\{v_1, v_2, v_3\}$ is LD. Then, we can get rid of one of the vectors in this set without changing the span. (So for example if $2v_1 + v_2 - 3v_3 = 0$ then we can get rid of v_2 since $v_2 = -2v_1 + 3v_3$,
So $V = \text{span}\{v_1, v_2, v_3\} = \text{span}\{v_1, v_3\}$) However, since $\dim(V) = 3$, any spanning set for V has to have 3 three vectors. This a contradiction so $\{v_1, v_2, v_3\}$ must be LI.
So it is also a basis.

Example Determine a basis for 3×3 skew-symmetric ($A^T = -A$) matrices.

$$S = \{ A \in M_3(\mathbb{R}) \mid A^T = -A \}$$

$$S = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \;\middle|\; a, b, c, \dots, g, h, i \in \mathbb{R} \right\}$$

$$A^T = -A: \quad \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{pmatrix}$$

$a = -a \Rightarrow a = 0$

$e = -e \Rightarrow e = 0$

$i = -i \Rightarrow i = 0$

$d = -b$

$g = -c$

$h = -f$

Blue part gives us the same equations

$$S = \left\{ \begin{pmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{pmatrix} \;\middle|\; b, c, f \in \mathbb{R} \right\}$$

$$b\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and b, c, f in $\mathbb{R}$ are free

$$\text{So } S = \text{span}\Bigg\{ \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{v_3} \Bigg\}$$

Exercise: Show v_1, v_2, v_3 are LI. So they form a basis for 3×3 skew-sym. matrices.

So moreover $\dim(S) = 3$. $\dim(M_3(\mathbb{R})) = 9$

Example
a) Determine a basis for 2×2 symmetric matrices. ($A^T = A$)
b) Find the dimension of the subspace.
c) Extend the basis you found in (a) to a basis for $M_2(\mathbb{R})$.

$$S = \{ A \in M_2(\mathbb{R}) \mid A^T = A \} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

~~$a = a$~~ ✓ ~~$d = d$~~ ✓

$b = c$

$$S = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$$

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{v_1} + b\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{v_2} + d\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{v_3}$$

So $S = \text{span}\{v_1, v_2, v_3\}$

again v_1, v_2, v_3 are LI.

Therefore $\{v_1, v_2, v_3\}$ is a basis for S and $\dim(S) = 3$

c) To extend $\{v_1, v_2, v_3\}$ to a basis for $M_2(\mathbb{R})$, we need to find a vector that is not in $\text{span}\{v_1, v_2, v_3\}$ and add it to the set.
Since $\text{span}\{v_1, v_2, v_3\}$ is all 2×2 sym. matrices, any (2×2) matrix that is not symmetric will work. e.g. $v_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Now, $\{v_1, v_2, v_3, v_4\}$ is a basis for $M_2(\mathbb{R})$.

$\dim(S) = 3$ $\qquad \dim(M_2(\mathbb{R})) = 4$

Example Find a basis for $S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x + 2y - z = 0 \\ -2x - y + z = 0 \end{array} \right\}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right) \xrightarrow{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array}\right) \to \begin{array}{l} x + 2y - z = 0 \\ y - z/3 = 0 \end{array}$$

leading ones

So z is a free variable

set $z = t \in \mathbb{R}$

$x + 2y - t = 0$

$y - t/3 = 0 \Rightarrow y = t/3$

$x + \frac{2t}{3} - t = 0 \Rightarrow x = t/3$

$$S = \{ (x, y, z) = (t/3, t/3, t) = t(\tfrac{1}{3}, \tfrac{1}{3}, 1) \mid t \in \mathbb{R} \}$$

$S = \text{span}\{(\frac{1}{3}, \frac{1}{3}, 1)\}$ and in fact $\{\underbrace{(\frac{1}{3}, \frac{1}{3}, 1)}_{v_1}\}$ is a basis for S.
so $\dim(S) = 1$.

Next extend it to a basis for $\mathbb{R}^3$.

Find $v_2 \notin \text{span}\{v_1\}$ $\qquad v_2 = (1, 0, 0) \notin \text{span}\{v_1\}$

~~$v_3 = (4/3, 1/3, 1) \notin \text{span}\{v_1\}$~~

Next, we find $v_3 \notin \text{span}\{v_1, v_2\} = \{ \underbrace{t(\frac{1}{3}, \frac{1}{3}, 1) + k(1, 0, 0)}_{(\frac{t}{3} + k,\ \frac{t}{3},\ t)} \mid t, k \in \mathbb{R} \}$

Claim
$v_3 = (0, 0, 1) \notin \text{span}\{v_1, v_2\}$

Pf If $v_3 \in \text{span}\{v_1, v_2\}$ then $(0, \boxed{0}, \boxed{1}) = (\frac{t}{3} + k, \boxed{t/3}, \boxed{t})$

$0 = t/3$

$1 = t$

Not possible

So therefore $v_3 \notin \text{span}\{v_1, v_2\}$

So $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

4.8 Row Space and Column Space

e.g.
$$A = \begin{pmatrix} 3 & 0 & 4 & -1 \\ -1 & 5 & 2 & 0 \\ 2 & 0 & 6 & -4 \end{pmatrix} \quad \in \mathbb{R}^4, \quad \in \mathbb{R}^3$$

$\text{colspace}(A) = \text{span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$

$\text{rowspace}(A) = \text{span}\{(3, 0, 4, -1), (-1, 5, 2, 0), (2, 0, 6, -4)\} \subseteq \mathbb{R}^4$

<u>Fact</u> $\dim(\text{colspace}(A)) = \dim(\text{rowspace}(A)) = \text{rank}(A)$

<u>Thm</u> a) To find a basis for <u>row space (A)</u>
- reduce it to row-echelon form B and take all non-zero <u>rows in B</u>

b) To find a basis for col space (A)
- reduce it to row-echelon form B and take <u>columns of A</u> that ended up having a leading 1 in B

$$A = \begin{pmatrix} 3 & 0 & 4 & -1 \\ -1 & 5 & 2 & 0 \\ 2 & 0 & 6 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 3 \\ -1 & 5 & 2 & 0 \\ 2 & 0 & 6 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & 10 & -10 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

leading ones

$\{(1, 0, -2, 3), (0, 1, 0, 3/5), (0, 0, 1, -1)\}$ is a basis for rowspace(A).

$\left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} \right\}$ is a basis for colspace (A).

AFTER CLASS DISCUSSION

<u>Thm</u> Say $\{v_1, v_2, \ldots, v_n\} \subseteq V$ and $\dim(V) = n$

Then this set being LI is equivalent to being a basis is equivalent to being a spanning set

$1, x-1, x^2-x, x^3-x^2, x^4-x^3$ is a spanning set for $P_4(\mathbb{R})$

span x span x^2 x^3 x^4 LI $\dim(P_4 \mathbb{R}) = 5$

V $\dim(V) = n$ $\quad S \subseteq V \sim \mathbb{R}^n, P_n(\mathbb{R})$ (S: subspace)

$M_{n \times m}(\mathbb{R})$