Questions

1) Say $\{v_1, v_2, v_3\}$ in V is LD (linearly dependent) and $v_4 \in V$. What can we say about LI/LD of

a) $\{v_1, v_2\}$ — It depends!

b) $\{v_1, v_2, v_3, v_4\}$? — LD

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ at least one of c_1, c_2, c_3 is not zero.

$c_1 v_1 + c_2 v_2 + c_3 v_3 + 0 v_4 = 0$ is a linear dependence relation v_1, v_2, v_3, v_4

4th

$\boxed{v_1 + v_3 = 0}$

$\{v_1, v_2\} \rightsquigarrow$ LI $\{v_1, v_3\} \rightsquigarrow$ LD

2) Say $\{v_1, v_2, v_3\}$ is LI. What about

a) $\{v_1, v_2\}$ — $\boxed{\text{LI}}$

b) $\{v_1, v_2, v_3, v_4\}$? — It depends

(a) If $\{v_1, v_2\}$ is LD by the prev. example "adding" a vector to the set results in a LD set. So $\{v_1, v_2, v_3\}$ has to be LD.

(b) v_4 might be a Lin. Combination of v_1, v_2, v_3 (Or $v_4 = 0$) $\Rightarrow$ LD

Or for example $v_1 = (1,0,0,0)$ $v_2 = (0,1,0,0)$ $v_3 = (0,0,1,0)$ $v_4 = (0,0,0,1)$ $\Rightarrow$ LI

$0v_1 + 0v_2 + 0v_3 + 1 \cdot v_4 = 0$

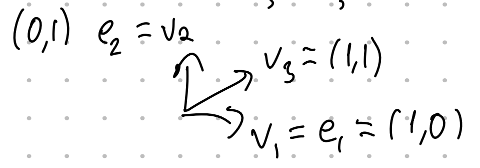
3) Say $\{v_1, v_2, v_3\}$ is a spanning set for V. What about

a) $\{v_1, v_2\}$ — Not necessarily

b) $\{v_1, v_2, v_3, v_4\}$? — Spanning ✓

(a) Say $v_1 = e_1 = (1,0,0)$, $v_2 = e_2 = (0,1,0)$, $v_3 = e_3 = (0,0,1)$

$\text{span}\{v_1, v_2\} = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$ does not span $\mathbb{R}^3$ "anymore".



$\text{Span}\{v_1, v_2\} = \{(a, b) \mid a, b \in \mathbb{R}\}$ still spans $\mathbb{R}^2$.

(b) Any element x in V can be written as $x = c_1 v_1 + c_2 v_2 + c_3 v_3 + 0 v_4$ for some c_1, c_2, c_3.

4) Say $\{v_1, v_2, v_3\}$ is not a spanning set for V.
Then $\{v_1, v_2\}$ is not a spanning set. $\{v_1, v_2, v_3, v_4\}$ may or may not be a spanning set.

Basis and Dimension

A subset S of a v.s. V is a basis if it is both LI and a spanning set for V.

If V has a basis consisting of n vectors, then

a) every spanning set for V has at least n vectors

b) every LI set in V has at most n vectors

c) By (a) and (b) every basis for V has exactly n vectors

In this case, we say that V is n-dimensional. $\dim(V) = n$.

Eg. $\{\underbrace{(1,0,0)}_{e_1}, \underbrace{(0,1,0)}_{e_2}, \underbrace{(0,0,1)}_{e_3}\}$ is the standard basis for $\mathbb{R}^3$.

In general, $e_1 = (1, 0, \cdots, 0)$, $e_2 (0, 1, 0, \cdots 0)$ - - -
$e_n = (0, \cdots, 0, 1)$ are the standard basis for $\mathbb{R}^n$

Thus, $\dim(\mathbb{R}^n) = n$

They are LI since $\det \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ 0 & \cdots & & & & 1 \end{pmatrix} = \det(I_n) = 1$

$\{\overset{x^0}{1}, x^1, x^2, \cdots, x^n\}$ is the standard basis for $P_n(\mathbb{R})$ = polynomials of order at most n.

eg. for $n=2$
LI: $W[1, x, x^2] = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$ $\boxed{\dim(P_n \mathbb{R}) = n+1}$

Example Verify that $\{\underbrace{1+x}_{p_1}, \underbrace{2-2x}_{p_2}, \underbrace{1+x^2}_{p_3}\}$ is a basis for $P_2(\mathbb{R})$.

We have to check LI: $W[p_1, p_2, p_3] = \begin{vmatrix} 1+x & 2-2x & 1+x^2 \\ 1 & -2 & 2x \\ 0 & 0 & 2 \end{vmatrix}$

$$= 2\begin{vmatrix} 1+x & 2-2x \\ 1 & -2 \end{vmatrix} = 2\left((1+x)(-2)-(1)(2-2x)\right) = 2\left(-2-\cancel{2x}-2+\cancel{2x}\right)$$
$$= -8 \neq 0$$

so $\{p_1, p_2, p_3\}$ is LI.

We need to verify they form a spanning set.

Say p_1, p_2, p_3 is not a spanning set for $P_2(\mathbb{R}) = V$. Then there is a 4^{th} polynomial p_4 in V such that p_1, p_2, p_3, p_4 are LI. However $\dim(P_2(\mathbb{R})) = 3$, it cannot have a LI set with 4 elements. This is a contradiction and thus, $\{p_1, p_2, p_3\}$ is a spanning set for $P_2(\mathbb{R})$.

Prop If V is n dimensional and $\{v_1, v_2, \dots, v_n\}$ is LI. Then $\{v_1, v_2, \dots, v_n\}$ is also a spanning set so In particular it is a basis.

Example Determine a subset of $\{(2,-6), (-1,3), (0,2)\}$ that is a basis for $\mathbb{R}^2$.

$$\{\underset{v_1}{(2,-6)}, \underset{v_2}{(-1,3)}, \underset{v_3}{(0,2)}\}$$

Since $\dim(\mathbb{R}^2) = 2$ and the set has 3 elements it cannot be LI. Notice that $v_1 = -2v_2$. (So $v_1 + 2v_2 = 0$)

This means I can get rid of v_1 (or v_2) without changing the span of the resulting set.

$\{v_2, v_3\} = \{(-1,3), (0,2)\}$. So we have 2 vectors in $\mathbb{R}^2$, they are LI if and only if $\det(v_2 \; v_3) \neq 0$

$$\det\begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} = -2 \neq 0 \quad \text{So } \{v_2, v_3\} \text{ is LI.}$$

Now we have 2 LI vectors in a 2-dimensional vector space so they form a spanning set by the prop. above.

Therefore $\{v_2, v_3\}$ is a basis for $\mathbb{R}^2$.