Recall: $\{v_1, v_2, \dots, v_k\}$ is LD ← linearly dependent if there are scalar $c_1, c_2, \dots c_k$ not all zero such that $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0$. such a (non-trivial) relation is called a linear dependence relation.

$\{\underset{v_1}{(1,2)}, \underset{v_2}{(1,0)}, \overset{v_3}{(3,-2)}\} \subseteq \mathbb{R}^2$

$c_1(1,2) + c_2(1,0) + c_3(3,-2) = (0,0)$

$(c_1+c_2+3c_3, 2c_1 - 2c_3) = (0,0)$ → $\boxed{\begin{matrix} c_1 + c_2 + 3c_3 = 0 \\ 2c_1 \quad 0 - 2c_3 = 0 \end{matrix}}$

$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -2 & -8 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right)$

leading ones

no leading one ⇓ c_3 free variable

$c_1 + c_2 + 3c_3 = 0$ set $c_3 = t$

$c_2 + 4c_3 = 0 \to c_2 = -4t$

$c_1 - 4t + 3t = 0 \quad c_1 = t$

$(c_1, c_2, c_3) = (t, -4t, t) = \boxed{t(1,-4,1)}$ t is scalar

plug in $t=1$ $(c_1, c_2, c_3) = (1,-4,1)$ gives us a linear dep. relation

$\boxed{0 = 1v_1 - 4v_2 + 1v_3 = (1,2) - 4(1,0) + (3,-2)}$

Note we could have asked find a lin. dep. relation for the polynomials

$v_1 = 1 + 2x \quad v_2 = 1 = 1 + 0x \quad v_3 = 3 - 2x$. $P_1(\mathbb{R}) = \{a + bx \mid a, b \in \mathbb{R}\}$

$c_1v_1 + c_2v_2 + c_3v_3 = 0 = 0 + 0x$

Example let $f_1(x) = 1$, $f_2(x) = 2\sin^2 x$, $f_3(x) = -5\cos^2 x$ are L.D. since

$f_1(x) = 1 = \sin^2(x) + \cos^2(x) = \frac{1}{2}f_2(x) - \frac{1}{5}f_3(x)$

$1 \cdot f_1(x) - \frac{1}{2}f_2(x) + \frac{1}{5}f_3(x) = 0$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

"Shortcut" of determining when $\{v_1, v_2, \dots v_k\}$ in $\mathbb{R}^n$ is LD or not.

Long-way:

$c_1v_1 + \dots + c_kv_k = 0$
find a nontrivial solution as above

$\underbrace{(v_1 \; v_2 \cdots v_k)}_{A_{n\times k}} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = 0$

no free variable

rank $A = k \Rightarrow$ LI

rank $A < k \Rightarrow$ LD

free variable

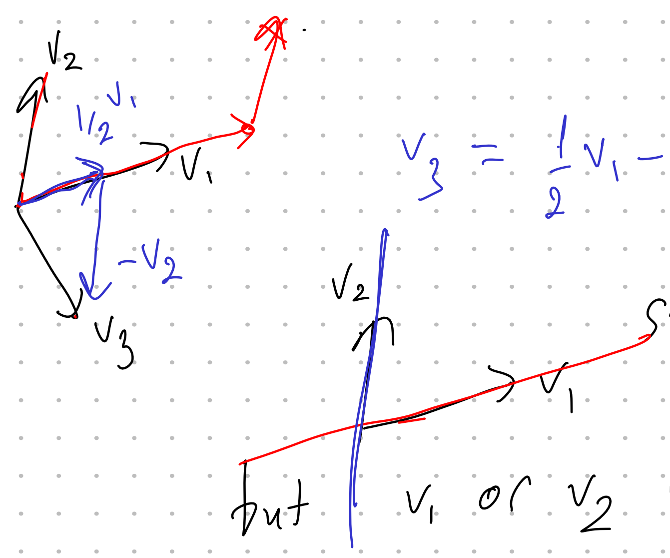
rank $A = \#$ leading ones in any row-echelon form of

Since rank $A_{n\times k} \le n, k$

Shortcut I: So in particular if $k = \#$ of vectors $> n \Rightarrow$ LD

II: When $n = k$, A is a square matrix rank$(A) < n^k \iff \det(A) = 0$ ⇓ LD

e.g. in $\mathbb{R}^2$



$v_3 = \frac{1}{2}v_1 - v_2$

spans $\mathbb{R}^2$

but v_1 or v_2 by itself only spans a line

So v_1 and v_2 are LI.

Wronskian $\quad W[f_1, f_2, \cdots, f_k]$

$$= \det \begin{pmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ f_1'' & f_2'' & \cdots & f_k'' \\ f_1^{(k-1)} & f_2^{(k-1)} & & f_k^{(k-1)} \end{pmatrix}_{k \times k}$$

Fact: If $W[f_1 \cdots f_k](x_0) \neq 0$ for some $x_0 \in I$, $\{f_1, \dots, f_k\}$ is LI.

Example: $f_1(x) = \sin x \qquad f_2(x) = \cos x \qquad x \in \mathbb{R}$

$$W[f_1, f_2](x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

So f_1 and f_2 LI.

Example: $f_1(x) = x, \quad f_2(x) = x^{-2}, \quad f_3(x) = x^{-4} \qquad \boxed{x > 0}$

$$W[f_1, f_2, f_3](x) = \left|\begin{matrix} x & x^{-2} & x^{-4} \\ 1 & -2x^{-3} & -4x^{-5} \\ 0 & 6x^{-4} & 20x^{-6} \end{matrix}\right| \begin{matrix} x & x^{-2} \\ 1 & -2x^{-3} \\ 0 & 6x^{-4} \end{matrix}$$

$$= \left(-40x^{-8} + 0 + 6x^{-8}\right) - \left(0 - 24x^{-8} + 20x^{-8}\right)$$

$$= -34x^{-8} + 4x^{-8} = -30x^{-8}$$

for $x = 1$, $W[f_1, f_2, f_3](1) = -30(1^{-8}) = -30 \neq 0$

Thus, $\{f_1, f_2, f_3\}$ are LI.

- If $W[f_1, \dots f_k] = 0$ for all x NO CONCLUSION

Example: Show that $f_1(x) = x^3 \qquad f_2(x) = \begin{cases} 2x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

are linearly indep.

Pf: Say $c_1, c_2 \in \mathbb{R}$ such that $\boxed{c_1 f_1 + c_2 f_2 = 0}$

$c_1 f_1(x) + c_2 f_2(x) = 0$ for all $x \in \mathbb{R}$.

So in particular for $x = 1$, we have $c_1 f_1(1) + c_2 f_2(1) = 0$

$x = -1$, $\quad c_1 f_1(-1) + c_2 f_2(-1) = 0 \qquad c_1 + 2c_2 = 0$

$-c_1 + c_2 = 0 \rightarrow c_1 = c_2 \leftrightarrow c_1 = -2c_2$

$c_2 = -2c_2$

$3c_2 = 0 \qquad c_2 = 0$

$c_1 = 0$

So we are forced to choose $c_1 = 0$ and $c_2 = 0$. Therefore f_1 and f_2 are LI

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e.g.

$$A = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad \text{then} \quad \det \underbrace{\begin{pmatrix} r_1 \\ k r_2 \\ r_3 \end{pmatrix}}_{B} = k \det(A)$$

B is obtained from A by multiplying the second row by k, and $\det(B) = 5$. What is $\det(A)$?

$$5 = \det(B) = k \det(A) \qquad \det(A) = \frac{5}{k}$$