Last time: If $A \in M_{m\times n}(\mathbb{R})$, $S = \{ x \mid Ax = 0 \}$ is a subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$), which we denote by nullspace(A).

Example: Find the nullspace of

$$A = \begin{pmatrix} 2 & 1 & -6 \\ -1 & -1/2 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to solve $Ax = 0$.

$$A \sim \begin{pmatrix} 1 & 1/2 & -3 \\ -1 & -1/2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 + \tfrac{1}{2}x_2 - 3x_3 = 0$$

x_1 is bound var.
x_2 and x_3 are free.

Set $x_2 = u$ and $x_3 = v \in \mathbb{R}$.

then $x_1 = -\tfrac{1}{2}u + 3v$.

$$\text{nullspace}(A) = \{ (-\tfrac{1}{2}u + 3v, u, v) \mid u, v \in \mathbb{R} \}$$

$$(-\tfrac{1}{2}u+3v, u, v) = (-\tfrac{1}{2}u, u, 0) + (3v, 0, v)$$
$$= u(-\tfrac{1}{2}, 1, 0) + v(3, 0, 1)$$

In this case we say that nullspace(A) is spanned (or generated) by $(-\tfrac{1}{2}, 1, 0)$ and $(3, 0, 1)$.

Thm: The set of all solutions to the homogeneous linear diff. eq.

$$y'' + a_1(x)y' + a_2(x)y = 0$$

on an interval I is a subspace of all real-valued functions on I. (In particular, the solution space is a vector space.)

Pf: Exercise

Example: The set of all solutions to $y'' - 3y' + 2y = 0$ is given by $S = \{ ce^t + de^{2t} \mid c, d \in \mathbb{R} \}$

Note that S is spanned by e^t and e^{2t}

4.4 Spanning Sets

Given vectors $v_1, v_2, \dots, v_k$ in a v.s. V, and scalars $c_1, c_2, \dots, c_k$ the expression of the form

$$c_1v_1 + c_2v_2 + \dots + c_kv_k$$ is called a linear combination of $v_1, v_2, \dots, v_k$.

Def^n: If every vector in a v.s. V can be written as a linear combination of $v_1, v_2, \dots, v_k$ we say that V is spanned or generated by $v_1, v_2, \dots, v_k$ and call the set of vectors $\{v_1, v_2, \dots, v_k\}$ a spanning set for V. In this case we also say that $\{v_1, v_2, \dots, v_k\}$ spans V.

Example: $\mathbb{R}^2$ is spanned by $v_1 = (1,1)$ and $v_2 = (2,-1)$.

So given $(a,b) \in \mathbb{R}^2$, we should be able to come up with c_1 and c_2 so that

$$(a,b) = c_1v_1 + c_2v_2$$
$$= c_1(1,1) + c_2(2,-1)$$
$$= (c_1, c_1) + (2c_2, -c_2)$$
$$= (c_1 + 2c_2, c_1 - c_2)$$

So
$$c_1 + 2c_2 = a$$
$$c_1 - c_2 = b$$
$$\longrightarrow A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$\det(A) = -1 - 2 = -3 \neq 0$. So A is invertible and therefore there is a unique solution to this system.

In fact, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

So
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{3}a + \frac{2}{3}b \\ \frac{1}{3}a - \frac{1}{3}b \end{pmatrix}$$

$$(a,b) = \underbrace{\left(\tfrac{1}{3}a + \tfrac{2}{3}b\right)}_{\text{scalar}} \underbrace{(1,1)}_{\text{vector}} + \underbrace{\left(\tfrac{1}{3}a - \tfrac{1}{3}b\right)}_{\text{scalar}} \underbrace{(2,-1)}_{\text{vector}}$$

Example: Determine whether $\mathbb{R}^3$ is spanned by

$v_1 = (1, -3, 6)$
$v_2 = (1, -4, 2)$
$v_3 = (-2, 10, 4)$.

Given $(a,b,c) \in \mathbb{R}^3$, can we come up with c_1, c_2, c_3 so that $(a,b,c) = c_1v_1 + c_2v_2 + c_3v_3$?

$$c_1(1,-3,6) + c_2(1,-4,2) + c_3(-2,10,4) = (a,b,c)$$

1st components:
$$c_1 + c_2 - 2c_3 = a$$
2nd:
$$-3c_1 - 4c_2 + 10c_3 = b$$
3rd:
$$6c_1 + 2c_2 + 4c_3 = c$$

$$A^{\#} = \left(\begin{array}{ccc|c} 1 & 1 & -2 & a \\ -3 & -4 & 10 & b \\ 6 & 2 & 4 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & a \\ 0 & -1 & 4 & 3a+b \\ 0 & -4 & 16 & -6a+c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & a \\ 0 & 1 & -4 & -3a-b \\ 0 & -4 & 16 & -6a+c \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & a \\ 0 & 1 & -4 & -3a-b \\ 0 & 0 & 0 & -18a-4b+c \end{array}\right)$$

If $-18a-4b+c=0$, then this system has a solution since in that case $\text{rank}(A)=\text{rank}(A^{\#})$. However in general $\text{rank}(A) < \text{rank}(A^{\#})$.

So there are some vectors in $\mathbb{R}^3$ which are not linear combinations of v_1, v_2, v_3. So v_1, v_2, v_3 do not span $\mathbb{R}^3$.

Note: set $A = (-18 \quad -4 \quad 1)$ then $\text{nullspace}(A) = \{(a,b,c) \mid -18a-4b+c=0\}$ is a subspace of $\mathbb{R}^3$ and the same computations as above show that v_1, v_2, v_3 span $\text{nullspace}(A)$.

Thm Let $v_1, v_2, \dots, v_k$ be vectors in $\mathbb{R}^n$. Then $\{v_1, \dots, v_k\}$ spans $\mathbb{R}^n$ if and only if for the matrix $A = (v_1 \ v_2 \cdots v_k)$ the linear system $Ac = v$ is consistent for every $v \in \mathbb{R}^n$.

Example $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ spans $M_{2\times 2}(\mathbb{R})$:

Given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times2}(\mathbb{R})$ we want to find c_1, c_2, c_3, c_4 s.t.

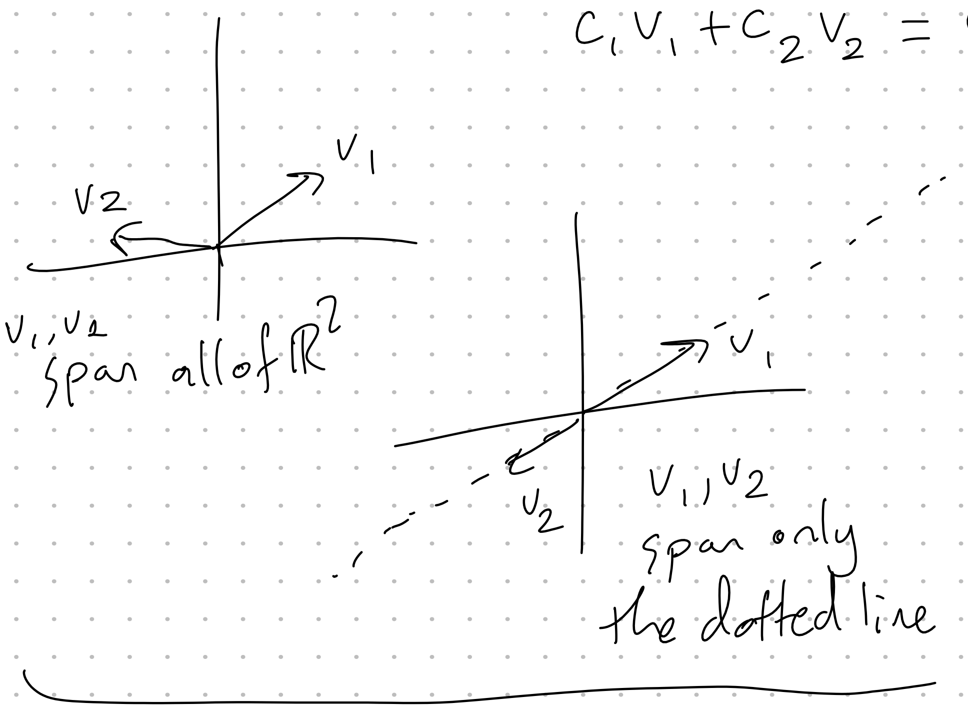
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_3 & c_3 \\ c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_4 & c_4 \\ c_4 & c_4 \end{pmatrix}$$

$$\begin{aligned} c_1+c_2+c_3+c_4 &= a \\ c_2+c_3+c_4 &= b \\ c_3+c_4 &= c \\ c_4 &= d \end{aligned}$$

This system has a (unique) solution. So in particular, $\{A_1, A_2, A_3, A_4\}$ is a spanning set for $M_{2\times2}(\mathbb{R})$.

$c_1v_1 + c_2v_2 = (a,b) \in \mathbb{R}^2$



Let $v_1, v_2, \dots, v_k$ be vectors in a v.s. V. Forming all possible linear combinations of $v_1, v_2, \dots, v_k$ generates a subset of V called the linear span of $\{v_1, \dots, v_k\}$ denoted $\text{span}\{v_1, \dots, v_k\}$.

Thm Let $v_1 \dots v_k$ be vectors in a v.s. V. Then $\text{span}\{v_1, \dots, v_k\}$ is a subspace of V.

Example If $V = \mathbb{R}^2$, $v_1 = (-2, 3)$ determine $\text{span}\{v_1\}$.

$$\begin{aligned} \text{span}\{v_1\} &= \{v \in \mathbb{R}^2 \mid v = c_1v_1, \ c_1 \in \mathbb{R}\} \\ &= \{v \in \mathbb{R}^2 \mid v = c_1(-2,3), \ c_1 \in \mathbb{R}\} \\ &= \{v \in \mathbb{R}^2 \mid v = (-2c_1, 3c_1), \ c_1 \in \mathbb{R}\} \\ &= \{(-2c_1, 3c_1) \mid c_1 \in \mathbb{R}\} \end{aligned}$$

Example Let $V = \mathbb{R}^3$, $v_1 = (1,0,1)$, $v_2 = (0,1,1)$. Determine the $\text{span}\{v_1, v_2\}$

$$\begin{aligned} &= \{v \in \mathbb{R}^3 \mid v = c_1v_1 + c_2v_2, \ c_1, c_2 \in \mathbb{R}\} \\ &= \{v \in \mathbb{R}^3 \mid v = (c_1, 0, c_1) + (0, c_2, c_2), \ c_1, c_2 \in \mathbb{R}\} \\ &= \{(c_1, c_2, c_1+c_2) \mid c_1, c_2 \in \mathbb{R}\} \end{aligned}$$

4.5 Linear Dependence and Linear Independence

Example The sets

$$S_1 = \{(1,0), (0,3)\}, \quad S_2 = \{(1,1), (0,1)\}, \quad S_3 = \{(1,1), (0,1), (5,4)\}$$

are all spanning sets for $\mathbb{R}^2$. Is there a "best" spanning set?

Clearly, S_2 is contained in S_3 and S_2 is smaller. In other words S_3 has a redundant element! S_1 and S_2 both have 2 elements. They are minimal spanning sets.