More Examples of Vector Spaces

6) $\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) \mid z_1, z_2, \ldots z_n \in \mathbb{C}\}$

$(z_1, z_2 \ldots z_n) + (w_1, w_2, \ldots, w_n)$
$= (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$

$k(z_1, z_2, \ldots z_n) = (kz_1, kz_2, \ldots, kz_n)$

where $k \in \mathbb{C} \rightsquigarrow \mathbb{C}^n$ is a complex v.s.
$(k \in \mathbb{R}) \longrightarrow \mathbb{C}^n$ is a real v.s.

$u = (-6i, -3+2i) \in \mathbb{C}^2$,
$v = (-3i, \sqrt{2}, 0, -\pi) \in \mathbb{C}^4$

For $u = (3, 2+i)$ $\quad v = (10i, -3+i) \in \mathbb{C}^2$

Find $3u + iv = 3(3, 2+i) + i(10i, -3+i)$
$= (9, 6+3i) + (-10, -1-3i)$
$= (-1, 5) \in \mathbb{C}^2$

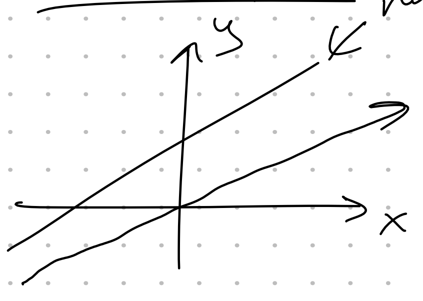
7) Let V be the set of all polynomials with real coefficients and of degree 2 or less, together with usual operations of polynomial addition, and multiplication of a polynomial by a real numbers. Then V is a <u>vector space</u>.

let $p(x) = 5 + 3x^2$, $\quad q(x) = 2x + x^2$

Then $6p(x) - 3q(x) = 6(5 + 3x^2) - 3(2x + x^2)$
$= 30 + 18x^2 - 6x - 3x^2$
$= 30 - 6x + 15x^2 \in V$

Say $r(x) = x^2 + 2 \qquad s(x) = -x^2$
$r(x) + s(x) = \cancel{x^2} + 2 \cancel{-x^2} = 2 \in V$

4.3 Subspaces



not a subspace

a subspace of $\mathbb{R}^2$ is a line that passes through the origin.

Def^n Let S be a non-empty subset of a vector space V. If S itself is a vectorspace under <u>the same operations</u> of addition and scalar multiplication as used in V, then we say that <u>S is a subspace</u> of V.

Thm Let S be a subset of a vector space V. Then S is a subspace if and only if

1) $0 \in S$
2) S is closed under addition
3) S is closed under scalar multiplication.

Further Properties

Let V be a vector space over $\mathbb{F}$. Then

1) The zero vector is unique.
2) $\underset{\text{scalar}}{0}\, v = \underset{\text{vector}}{0}$ for all $v \in V$.
3) $k\,0 = 0$ for all $k \in \mathbb{F}$.
4) The additive inverse $-x$ of x is unique for all $x \in V$.
5) For all $v \in V$, $-v = (-1)v$
6) If $k \in \mathbb{F}$, $v \in V$ such that $kv = 0$, then either $k = 0$ or $v = 0$.

Example The set of solutions to the system

$x - 3y + 3z = 0$
$-x + 2y + z = 0$

is a subspace of $\mathbb{R}^3$.

1) $(x, y, z) = (0, 0, 0)$ satisfies the system so $0 \in S$.

Next we have to verify (2) and (3).

$\begin{pmatrix} 1 & -3 & 3 \\ -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 3 \\ 0 & -1 & 4 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -4 \end{pmatrix} \to \begin{array}{l} x - 3y + 3z = 0 \\ y - 4z = 0 \end{array}$

x, y bound; z is a free var. Set $z = t \in \mathbb{R}$

$x - 3y + 3t = 0$
$y - 4t = 0 \qquad y = 4t$

$x - 3(4t) + 3t = 0$
$x = 9t$

$S = \{(9t, 4t, t) \mid t \in \mathbb{R}\}$

2) Say $u, v \in S$. Then $u = (9r, 4r, r)$ and $v = (9s, 4s, s)$ for some $r, s \in \mathbb{R}$.

$u + v = (9r + 9s, 4r + 4s, s + r) = (9(r+s), 4(r+s), r+s)$
$\in S$ since $r + s \in \mathbb{R}$

3) $k \in \mathbb{R}$ $\quad (9t, 4t, t) \in S$

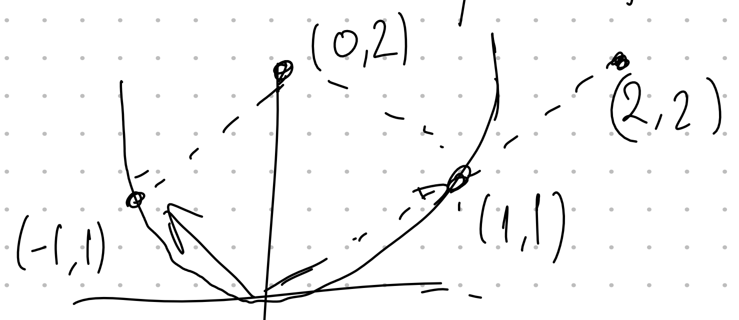
$k(9t, 4t, t) = (9(kt), 4(kt), kt)$
$\in S$ since $kt \in \mathbb{R}$.

So by the Theorem, S is a subspace.

Non-example $S = \{x \in \mathbb{R}^2 \mid x = (r, -3r+1),\ r \in \mathbb{R}\}$

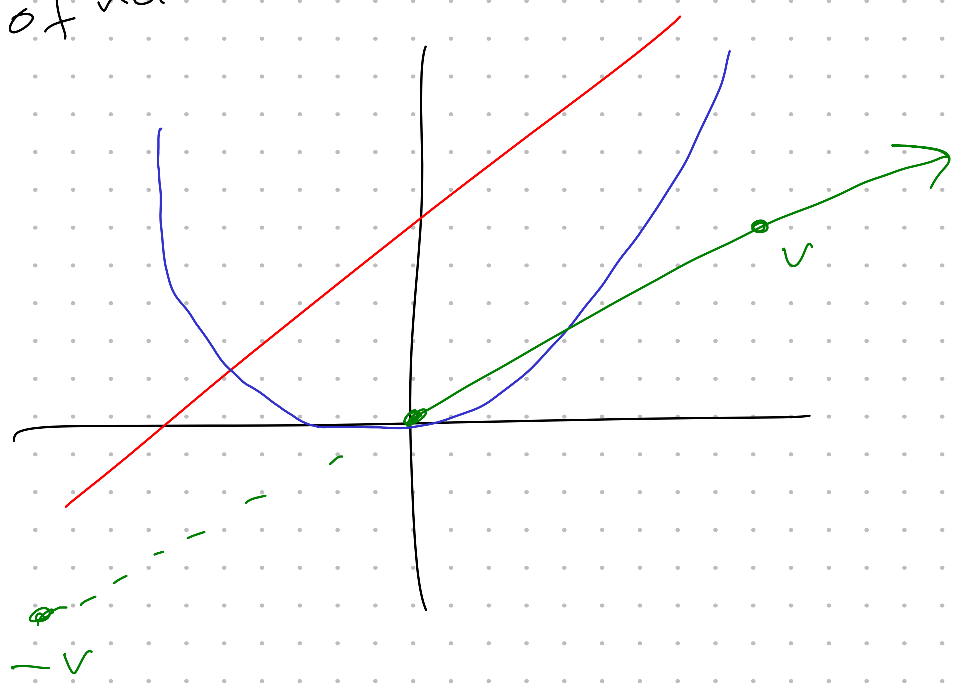
is not a subspace of $\mathbb{R}^2$ because $0 \notin S$. [Assume on the contrary that $0 \in S$. Then there is $r \in \mathbb{R}$ such that $(0, 0) = (r, -3r + 1)$ but the 1st component $\Rightarrow r = 0$
$\Rightarrow$ the 2nd component $-3r + 1 = 1 \neq 0$ and that's a contradiction]

Non-example $S = \{(x, x^2) \mid x \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^2$.



S is not closed under scalar multiplication: $(1,1) \in S$.
However, $2(1,1) = (2,2) \notin S$.

Pictures of non-examples



Example let S be the set of all skew-symmetric $n \times n$ matrices. Then S is a subspace of $M_n(\mathbb{R})$ = $n \times n$ matrices over $\mathbb{R}$

Note that $S = \{A \in M_n(\mathbb{R}) \mid A^T = -A\}$

1) Clearly, $0^T = -0$ so $0 \in S$.

2) Say $A, B \in S$. Then
$(A+B)^T = A^T + B^T = -A - B = -(A+B)$

3) Given $A \in S$, $k \in \mathbb{R}$,

$$(kA)^T = kA^T = k(-A) = k(-1)A = -kA = -(kA)$$

So S is a subspace by the theorem. ✓

Example let V be the set of all real valued functions on an interval $[a,b]$. let S denote the set of all functions which satisfy $f(a) = f(b)$ i.e.
$S = \{f \in V \mid f(a) = f(b)\}$.

1) Clearly, $0(x) = 0$ for all $x \in [a,b]$ is in S as $0(a) = 0 = 0(b)$

2) Given $f, g \in S$,
$(f+g)(a) = f(a) + g(a) = f(b) + g(b) = (f+g)(b)$
so $f+g \in S$.

3) Given $f \in S$, $k \in \mathbb{R}$,
$(kf)(a) = kf(a) = kf(b) = (kf)(b)$ so $kf \in S$. ✓

Thm $S = \{0\} \subseteq V$ ("contained in") is a subspace of V.

Thm Let $A \in M_{m \times n}(\mathbb{R})$ or $\mathbb{C}$ ← set of $m \times n$ matrices.
The solution set of $Ax = 0$ is a subspace of $\mathbb{C}^n$ (or $\mathbb{R}^n$)

Thus,
$\text{nullspace}(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$

Def^n let A be an $m \times n$ matrix, the solution set to the homogeneous system $Ax = 0$ is called the nullspace of A, denoted nullspace(A).